Multiple Regression Analysis : Further Issues

1 Data scaling on OLS statistics

When we change the unit of measurment of a variable, the value of estimators would chance accordingly. For example

$$\widehat{bweght}_g = \widehat{\beta}_0 + \widehat{\beta}_1 cigs + \widehat{\beta}_2 fa\min c,$$

where
$bwght$ = child birth weight, in grams.
$cigs$ = number of cigarettes smoked by the mother while pregnant, per day.
$fa\min c$ = annual family income, in thousands of dollars.

- What if we use bweght in kilograms ??

1 kg. = 1,000 g.

$$\widehat{bweght}_{kg} = \frac{\widehat{bweght}_g}{1{,}000} = \frac{\hat{\beta}_0}{1{,}000} + \frac{\hat{\beta}_1}{1{,}000} cigs + \frac{\hat{\beta}_2}{1{,}000} faminc$$

$$= \hat{\alpha}_0 + \hat{\alpha}_1 cigs + \hat{\alpha}_2 faminc.$$

$$\rightarrow \hat{\alpha}_0 = \frac{\hat{\beta}_0}{1{,}000}, \quad \hat{\alpha}_1 = \frac{\hat{\beta}_1}{1{,}000}, \quad \hat{\alpha}_2 = \frac{\hat{\beta}_2}{1{,}000}$$

- What if we use faminc in USD (instead of 1,000 USD)

$$bweght_g = \hat{\beta}_0 + \hat{\beta}_1 cigs + \frac{\hat{\beta}_2}{1{,}000} faminc_{USD}$$

$$= \hat{\beta}_0 + \hat{\beta}_1 cigs + \hat{\theta}_2 faminc_{USD}$$

The value of this variable is going to be 1,000 times larger than faminc

$$\rightarrow \hat{\theta}_2 = \frac{\hat{\beta}_2}{1{,}000}$$

in other words $\hat{\theta}_2$ = impact of 1 USD↑ in income

$\hat{\beta}_2$ = impact of 1,000 USD↑ in income

- What if we use bwght in Kg & income in THB

$$bweght_{kg} = \frac{\hat{\beta}_0}{1000} + \frac{\hat{\beta}_1}{1{,}000} cigs + \frac{\left(\frac{\hat{\beta}_2}{1{,}000}\right)}{30{,}000} faminc_{THB}$$

This value is going to be 30,000 times more than faminc.

$\frac{d\ln x}{dx} = \frac{1}{x} \rightarrow d\ln x = \frac{1}{x}dx$

2 More on functional forms

- Logarithmic Functional Form

y nonlog x log

usually mean natural log

$$\log(y) = \beta_0 + \beta_1 \log(x_1) + \beta_2 x_2 + u$$

$\Delta y = y_1 - y_2$, $\Delta x = x_{11} - x_{12}$

- $\beta_1 = \frac{d\log(y)}{d\log(x_1)} = \frac{\frac{1}{y}dy}{\frac{1}{x_1}dx_1} = \frac{\frac{1}{y}\Delta y}{\frac{1}{x_1}\Delta x} = \frac{100 \times \frac{1}{y}\Delta y}{100 \times \frac{1}{x_1}\Delta x_1} = \frac{\%\,y}{\%\,x}$

with the log y & log x format, the coefficient is going to be the elasticity! (x_1 elasticity of y)

(price) (demand)

- $\beta_2 = \frac{d\log(y)}{dx_2} = \frac{\frac{1}{y}dy}{dx_2} = \frac{\frac{1}{y}\Delta y}{\Delta x_2}$

→ if we want the upper term to be % change, then

$100\beta_2 = \frac{100\frac{1}{y}\Delta y}{\Delta x_2}$

$100\beta_2 = \frac{\%\,\Delta y}{\Delta x_2}$

$100\beta_2$ = % Δ in Y given that x_2 increases by 1 unit.

- Models with Quadratics (squares)

→ Capture increasing / decreasing marginal effects (slope of the relationship between X & Y is not constant.)

COVID-19 example.

Y (# of cases) → $y = \beta_0 + \beta_1 x + \beta_2 x^2 + u$

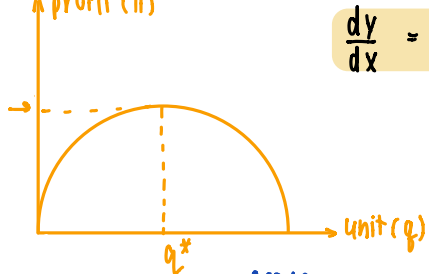
$\frac{dy}{dx} = \beta_1 + 2\beta_2 x$

(+) (+) days

Decreasing returns → $y = \beta_0 + \beta_1 x + \beta_2 x^2 + u$

$\frac{dy}{dx} = \beta_1 + 2\beta_2 x$

(+) (-)



Assume MC = 10

$\pi = (p - mc)q$; demand : $p = 100 - q$

$\pi = (100 - q - 10)q$

F.O.C. $\frac{\partial \pi}{\partial q} = 0 = 90 - 2q$

β_1 is positive

β_2 is (-)

Example : Effects of Pullution on Housing Prices

$$\log(price) = \beta_0 + \beta_1 \log(nox) + \beta_2 \log(dist) + \beta_3 rooms + \beta_4 room^2 + \beta_5 stratio + u$$

where
- $price$ = housing price
- nox = level of pollution
- $dist$ = distance from downtown
- $rooms$ = number of rooms
- $stratio$ = average student per teacher ratio

The estimation result is given by

In the US or many other countries, students can apply to schools in the area without having to take any test. so, the lower stratio, the better the school

```
regress  lprice lnox dist rooms rooms_sq stratio
```

Source	SS	df	MS		
Model	51.4933152	5	10.298663	Number of obs	= 506
Residual	33.0889098	500	.06617782	F(5, 500)	= 155.62
				Prob > F	= 0.0000
				R-squared	= 0.6088
				Adj R-squared	= 0.6049
Total	84.582225	505	.167489554	Root MSE	= .25725

log (price)

log (nox)

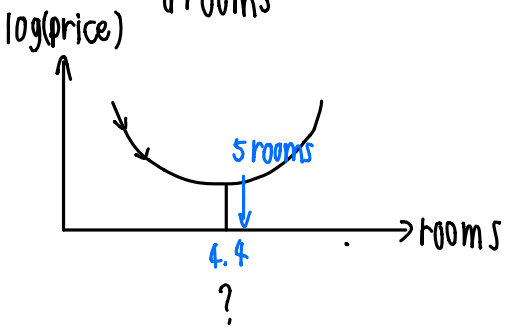
lprice	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
lnox (β_1)	-.9767545	.0995938	-9.81	0.000	-1.172429	-.7810806
dist (β_2)	-.0321972	.0094013	-3.42	0.001	-.050668	-.0137264
rooms (β_3)	-.5528032	.1612965	-3.43	0.001	-.8697056	-.2359007
rooms_sq (β_4)	.0624697	.0124867	5.00	0.000	.0379368	.0870025
stratio (β_5)	-.0486667	.0058131	-8.37	0.000	-.0600879	-.0372455
_cons	13.59154	.5650901	24.05	0.000	12.4813	14.70178

$|t| > 1.96$ all < 0.05

~all variable are significant

Consider the effect of "room"

$$\frac{d\log(price)}{d\,rooms} = \beta_3 + 2\beta_4\, rooms = -0.553 + 2(0.062)\cdot rooms$$



@ at how many rooms does 1 additional room has a positive impact on log (price)??

$$0 = -0.553 + 2(0.062)\cdot rooms$$
$$rooms = 4.4$$

Answer → at 4.4 rooms or more

at ~5 rooms or more.

What wold be the % change in price when the number of room increases from 5 to 6?

$$\frac{d\log(price)}{d\,room} = -0.553 + 2(0.062)\cdot rooms$$

$$\frac{100 \cdot \frac{1}{price}\, d(price)}{d\,room} = 100\,(-0.553 + 2(0.062)\cdot 5)$$

$$= 100 \times 0.067 = 6.7\%\ \text{increase.}$$

What about % change in price when # rooms increases from 5 to 7%. ??

$$\%\,\Delta price = 100(-0.553 + 2(0.062)\cdot 6) = 19.1\%$$

Total %Δ in price when # rooms ↑ from 5 to 7 is $6.7 + 19.1\% = 25.8\%$

3 Models with Interaction Terms → used when the impact of one variable depends on the value (level) of another variable

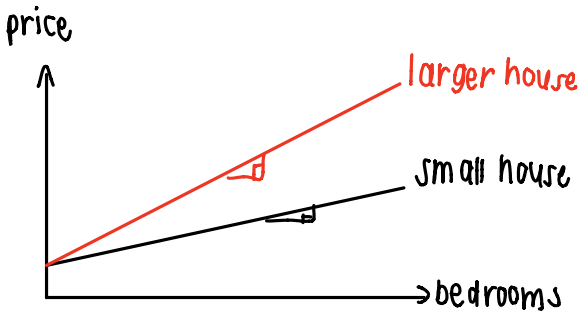
Consider

$$price = \beta_0 + \beta_1 sqrft + \beta_2 bdrms + \beta_3 sqrft \times bdrms + \beta_4 bthrms + u$$

x_1 x_2 x_3 ($x_1 \times x_2$) x_4

where

$price$ = housing price
$sqrft$ = house size (square feet)
$bdrms$ = number of bedrooms
$bthrms$ = number of bathrooms



$$\frac{\Delta price}{\Delta bdrms} = \beta_2 + \beta_3 sqrft$$

→ If $\beta_2 > 0$ then, an additional bedrooms would increase price more for a larger house!

4 More on the Goodness-of-Fit and Selection of Regressors

- Adding more regressors ALWAYS improve fit → R^2 always ↑

• But we lose the "degree of freedom"
(d.f. = free data point used to estimate the parameter)
↝ 1 data point is sacrified every time we estimate a parameter

• using R^2 would not punish "having too many regressors"

• we use adjusted_R^2 or R^{-2} when we want to punish adding too many regressors.

$$R^2 = 1 - \frac{SSR}{SST} = 1 - \frac{SSR/n}{SST/n}$$

$$\text{adj. } R^2 = 1 - \frac{SSR/(n-k-1)}{SST/(n-1)}$$

→ If d.f $= n-k-1 \downarrow$, $SSR/(n-k-1) \uparrow$, adj_$R^2 \downarrow$

Using adjusted R-squared to choose between non-nested models (one model is not a subset of another).
Consider Model 1

$$\begin{aligned} \widehat{salary} &= \underset{(223.90)}{830.63} + \underset{(0.0089)}{0.0163}sales + \underset{(11.08)}{19.63}roe \\ n &= 209, \quad R^2 = 0.029, \quad \bar{R}^2 = 0.020 \end{aligned}$$

Consider Model 2

$$\begin{aligned} \widehat{\log(salary)} &= \underset{(0.29)}{4.36} + \underset{(0.033)}{0.2751}\log(sales) + \underset{(0.004)}{0.0179}roe \\ n &= 209, \quad R^2 = 0.282, \quad \bar{R}^2 = 0.275 \end{aligned}$$

↳ 27.5% of variation in y is explained.
So, this model is better!

Multiple Regression Analysis with Qualitative Information:

1 Outline

info that represent quality
e.g. gender, season, month

- Describing qualitative information
- Using a single dummy independent variable
- Using dummy variables for multiple categories
- Interactions involving dummy variables
- A binary dependent variable (Y variable): The linear probability model

2 Describing Qualitative Information

- "Female" and "Married" are qualitative variable.
- We arbitarily assign a dummy variable to decribe them.

$$famale = \begin{cases} 1 & \text{if female} \\ 0 & \text{otherwise (or if male)} \end{cases}$$

$$married = \begin{cases} 1 & \text{if married} \\ 0 & \text{otherwise (of if single)} \end{cases}$$

TABLE 7.1

A Partial Listing of the Data in WAGE1.RAW

person	wage	educ	exper	female	married
1	3.10	11	2	1	0
2	3.24	12	22	1	1
3	3.00	11	2	0	0
4	6.00	8	44	0	1
5	5.30	12	7	0	1
⋮	⋮	⋮	⋮	⋮	⋮
525	11.56	16	5	0	1
526	3.50	14	5	1	0

3 Models with a single dummy independent variable

only effect value of intercept

Consider

$$wage = \beta_0 + \delta_0 female + \beta_1 educ + u. \quad (1)$$

where

$$famale = \begin{cases} 1 & \text{if female} \\ 0 & \text{otherwise (or if male)} \end{cases}$$

In this case, the δ_0 notation is used to highlight the interpretation of the parameters multiplying dummy variables. In other cases, we can use any notation that is the most convenient.

① $E(wage \mid female, educ) = E(\beta_0 + \delta_0 female + \beta_1 educ + u \mid female, educ)$

$= \beta_0 + \delta_0 female + \beta_1 educ + E(u \mid female, educ)$

$= \beta_0 + \delta_0 female + \beta_0 educ$ (assump. MLR1-4 ndds)

② thus

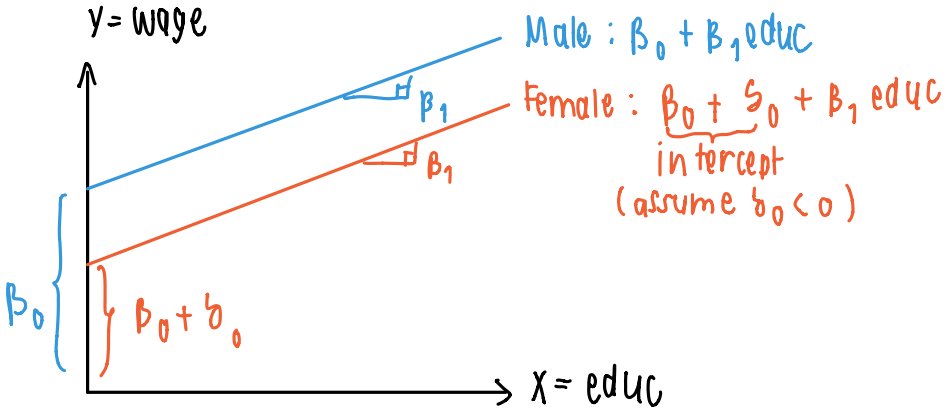
♀ : $E(wage \mid female=1, educ) = \beta_0 + \delta_0(1) + \beta_1 educ = \beta_0 + \delta_0 + \beta_1 educ$

♂ : $E(wage \mid female=0, educ) = \beta_0 + \delta_0(0) + \beta_1 educ = \beta_0 + \beta_1 educ$

$\delta_0 = E(wage \mid female=1, educ) - E(wage \mid female=0, educ)$

or $\delta_0 = E(wage \mid female, educ) - E(wage \mid male, educ)$

* given the same value of educ (same education level) δ_0 is the difference in the expected wage of females and males.



→ By the way we model this regression function, "female" is going to give a constant impact on wage, regardless of the level of educ.

4 It is not possible to include all of the dummy alternatives in the same model (as long as there is an intercept in the model)

- If we include all alternatives of a dummy variable in the same model, we will face the "perfect collinearity" problem.

$$wage = \beta_0 X_0 + \delta_0 female + \beta_1 educ + \delta_0 male + u$$
$$(X_1) \qquad (X_2) \qquad (X_3)$$

intercept + 1

For example:

$$X_0 = X_1 + X_3$$
$$1 = female + male$$
$$female = male + 1$$

Id	female	male	X_0
1	1	0	1
2	1	0	1
3	1	0	1
4	0	1	1
.	0	1	1
.	0	1	1
.	0	1	1
.	1	0	1
99			

or

If there are "n" categories, we omit "1" category to avoid multicollinearity

$$1 = winter + spring + summer + fall$$
$$winter = 1 - spring - summer - fall$$

$$winter = \begin{cases} 1 & \text{if winter} \\ 0 & \text{otherwise} \end{cases}$$

$$spring = \begin{cases} 1 & \text{if spring} \\ 0 & \text{otherwise} \end{cases}$$

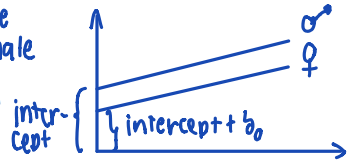
etc

id	winter	spring	summer	fall	X_0
1	1	0	0	0	1
2	1	0	0	0	1
3	0	1	0	0	1
4	0	1	0	0	1
.	0	0	1	0	1
.	0	0	1	0	1

- At least one alternative has to be dropped. We treat the dropped alternative as the "BASE GROUP" or "BASELINE" or "BENCHMARK GROUP".

↑ in this case, male

{ 1 if female, 0 if male } { 1 if male, 0 if female }




```
. regress lwage female male married educ exper
note: male omitted because of collinearity
```

Source	SS	df	MS	Number of obs =	526
Model	54.3265253	4	13.5816313	F(4, 521) =	75.27
Residual	94.0032262	521	.180428457	Prob > F =	0.0000
				R-squared =	0.3663
Total	148.329751	525	.28253286	Adj R-squared =	0.3614
				Root MSE =	.42477

lwage	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
female	-.3251146	.0377061	-8.62	0.000	-.3991892	-.25104
male	0	(omitted)				
married	.1380145	.0411197	3.36	0.001	.0572338	.2187953
educ	.0872644	.0071554	12.20	0.000	.0732075	.1013213
exper	.0076213	.0015314	4.98	0.000	.0046129	.0106297
_cons	.4690918	.1040575	4.51	0.000	.264668	.6735156

Female workers are expected to have less wage compare to male workers

5 Using dummy variables for multiple categories

Case 1 We can use many dummy variables in the same model

Consider a model which includes 2 dummy variables– *female* and *married*.

$$\log(wage) = \beta_0 + \delta_0 female + \delta_1 married + \beta_1 educ + \beta_2 \exp er + \beta_3 \exp er^2 + \beta_4 tenure + \beta_5 tenure^2 + u.$$

female: 1 if female, 0 if otherwise

married: 1 if married, 0 otherwise

```
regress lwage female married educ exper expersq tenure tenursq
```

Source	SS	df	MS
Model	65.6482326	7	9.37831895
Residual	82.6815188	518	.159616832
Total	148.329751	525	.28253286

Number of obs	= 526
F(7, 518)	= 58.76
Prob > F	= 0.0000
R-squared	= 0.4426
Adj R-squared	= 0.4351
Root MSE	= .39952

lwage	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
female	-.2901838	.0361121	-8.04	0.000	-.3611279	-.2192396
married	.0529219	.0407561	1.30	0.195	-.0271456	.1329894
educ	.0791547	.0068003	11.64	0.000	.0657952	.0925143
exper	.0269535	.0053258	5.06	0.000	.0164907	.0374163
expersq	-.0005399	.0001122	-4.81	0.000	-.0007603	-.0003196
tenure	.0312962	.0068482	4.57	0.000	.0178426	.0447499
tenursq	-.0005744	.0002347	-2.45	0.015	-.0010355	-.0001134
_cons	.4177837	.0988662	4.23	0.000	.2235557	.6120116

$\hat{\beta}$

2. δ_1 measure the impact of be married. (marriage premium) But since $|t| < 1.96$ or $p > 0.05$, we do not reject H_0 of no impact.

Comments:

1.) δ_0 measures the expected difference between female & male workers given the same marital status and other factors

$$\frac{\partial \log(wage)}{\partial female} = \frac{\frac{1}{wage} d wage}{\partial female} = -0.29$$

$$\frac{100 \cdot \frac{1}{wage} dwage}{\partial female} = 100 \cdot (-.29)$$

$$\frac{\%\Delta wage}{\partial female} = 29.02\%.$$

- female worker are expected to earn less than male workers by 29.02%, holding other factors the same.

	♀	♂
marr	marfem	mar male
sing	sing fem	sing male

Base case

Consider a model which includes dummy variables for each gender/marital status combination– *marrmale*, *marrfem* and *singfem*. (or singmale ← used as the basecase

$$\log(wage) = \beta_0 + \delta_0 marrmale + \delta_1 marrfem + \delta_3 singfem + \beta_1 educ + \beta_2 exper + \beta_3 exper^2 + \beta_4 tenure + \beta_5 tenure^2 + u. \quad (8.1)$$

```
regress lwage marrmale marrfem singfem educ exper expersq tenure tenursq
```

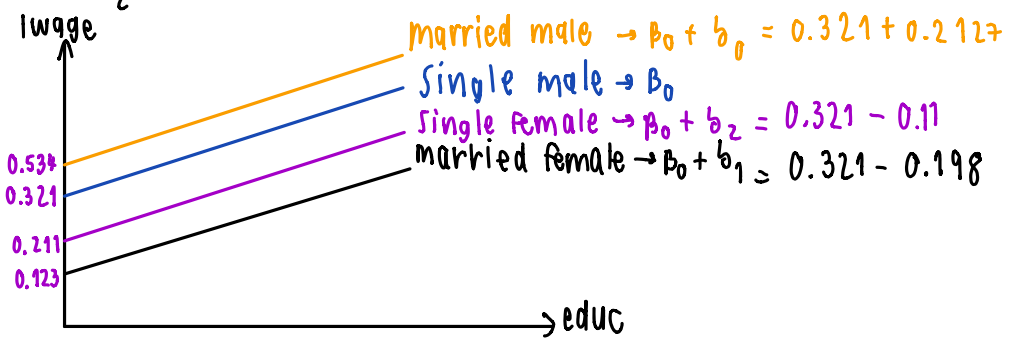
Source	SS	df	MS		
Model	68.3617623	8	8.54522029	Number of obs	= 526
Residual	79.9679891	517	.154676961	F(8, 517)	= 55.25
				Prob > F	= 0.0000
Total	148.329751	525	.28253286	R-squared	= 0.4609
				Adj R-squared	= 0.4525
				Root MSE	= .39329

lwage	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
δ_0 marrmale	.2126757	.0553572	3.84	0.000	.103923	.3214284
δ_1 marrfem	-.1982676	.0578355	-3.43	0.001	-.311889	-.0846462
δ_2 singfem	-.1103502	.0557421	-1.98	0.048	-.219859	-.0008414
$\hat{\beta}$ educ	.0789103	.0066945	11.79	0.000	.0657585	.092062
exper	.0268006	.0052428	5.11	0.000	.0165007	.0371005
expersq	-.0005352	.0001104	-4.85	0.000	-.0007522	-.0003183
tenure	.0290875	.006762	4.30	0.000	.0158031	.0423719
tenursq	-.0005331	.0002312	-2.31	0.022	-.0009874	-.0000789
_cons	.3213781	.100009	3.21	0.001	.1249041	.5178521

This regression is not the same as the previous one.
It uses "single male" as the base group.
(The previous one use male & single as 2 base groups.)

Comments:

- δ_0 measure the expected diff. in wage of married male as compared with single males, holding other factor constant.
- δ_1 measure the expected diff. in wage of married female as compared with single males, holding other factor constant.
- δ_2 ~ same rationale



Case 2 We can use dummy variables to represent multiple categories of a variable
Consider the relationship between law school rankings and starting salaries

$$\log(salary) = \beta_0 + \delta_0 top10 + \delta_1 r11\_25 + \delta_3 r26\_40 + \delta_4 r41\_60 + \beta_1 LSAT + \beta_2 GPA + \beta_3 \log(libvol) + \beta_4 \log(cost) + u.$$

where $top10$, $r11\_25$, $r26\_40$, $r41\_60$ would be equal to 1 when the variable $rank$ falls into the appropriate range.

** Rank below 60 would be the base case.

* In many cases the "range of value" serve as a better explanatory variables than the "value" itself.
e.g. age may explain the model better if split into generations
young 0-15
gen z 16-19
etc.

```
. regress lsalary top10 r11_25 r26_40 r41_60 LSAT GPA llibvol lcost
```

Source	SS	df	MS		
Model	9.16538532	8	1.14567316	Number of obs	= 136
Residual	1.2109665	127	.009535169	F(8, 127)	= 120.15
				Prob > F	= 0.0000
				R-squared	= 0.8833
				Adj R-squared	= 0.8759
Total	10.3763518	135	.076861865	Root MSE	= .09765

lsalary	Coef.	Std. Err.	t	P>|t|	[95% Conf.	Interval]
top10	.5393428	.053542	10.07	0.000	.4333927	.6452928
r11_25	.4716199	.0390921	12.06	0.000	.3942637	.548976
r26_40	.2790977	.0346972	8.04	0.000	.2104383	.3477571
r41_60	.182382	.0283098	6.44	0.000	.126362	.238402
LSAT	.0060482	.0034919	1.73	0.086	-.0008616	.012958
GPA	.1305893	.0818678	1.60	0.113	-.0314122	.2925908
llibvol	.0725522	.0289213	2.51	0.013	.0153221	.1297824
lcost	.0249169	.0283224	0.88	0.381	-.031128	.0809619
_cons	8.363103	.4457314	18.76	0.000	7.481081	9.245125

the baseline is ranking 61th and worse

Comments:

rank	top 10	r11-25	r 26-40
1	1	0	0
2		0	0
3		0	0
⋮		0	0
10			0
11	0		0
⋮	0		0
25	0		0
⋮		0	1
40		0	
⋮		0	

1) δ_0 measures the difference in expected log(salary) of a law-school graduate from a top 10 university compared to expected log(salary) of those who graduated from the school ranked 61th and worse

2) δ_1 ~ use the same rationale