EE320 (1/2017)

INTRODUCTORY MATHEMATICAL ECONOMICS

INTEGRATION AND ITS APPLICATION

Topics

- Terminology in Integration
- Indefinite Integration
- Basic Rules of Integration
- Definite Integration
- Improper Integrals

What is Integration?

- *Integration* is the inverse of differentiation.
- Formally, an integral is a function F(X) whose derivative is f(x):

$$F'(x) = f(x)$$

- This function F(x) can then be called an 'anti-derivative' of f(x).

Example:

- $f(x) = nx^{n-1}$ ➔ $F(x) = x^n$
- $f(x) = \frac{1}{x}$ ➔ F(x) = ln(x)

- The process of anti-differentiation is called integration.
- That is, to integrate a function f(x) is to find F(X) such that

$$F'(x) = f(x)$$

Indefinite Integrals

- If we integrate a function f(x) where values of x are not given, we have to *integrate without a limit* (i.e. *to find indefinite integral*).
- A symbol for integrating a function f(x) is:

$$\int f(x)dx = F(x) + C \Leftrightarrow F'(x) = f(x)$$

Where $\int$ is called the *integral sign*.

f(x) is called the *integrand*.

C is called the *constant of integration*.

dx indicates the variable involved in the integration.

- Note: A function does not have a unique integral.

Example: If $f(x) = 3x^2$, then $F(x) = x^3+1$ or x^3+7 or $x^3+ C$

Basic Rules of Integration

Rule I. Power rule: $\int x^n dx = \frac{1}{n+1} x^{n+1} + c, n \neq -1$

Rule II. Exponential rule: $\int e^{ax} dx = \frac{1}{a} e^{ax} + c$

Rule III. Logarithmic rule: $\int \frac{1}{x} dx = \ln|x| + c$

Rule IV: Integral of a sum

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

Rule V: Integral of a multiple

$$\int af(x) dx = a \int f(x) dx$$

Rules of Operations

Rule IIa:

$$\int a^{bx} dx = \frac{1}{b \ln a} a^{bx} + c$$

Rule IIb:

$$\int f(x) e^{f(x)} dx = e^{f(x)} + c$$

Rule IIIa:

$$\int \frac{f'(x)}{f(x)} dx = \ln\left| f(x) \right| + c, [f(x) \neq 0]$$

Examples

Find integrals of the following functions:

- $\int \frac{1}{x^3} dx = -\frac{1}{2x^2} + c$
- $\int \sqrt{x\sqrt{x\sqrt{x}}} dx = \int x^{7/8} dx = \frac{8}{15} x^{15/8} + c$
- $\int (3x^4 + 5x^2 - 2) dx = \frac{3}{5} x^5 + \frac{5}{3} x^3 - 2x + c$
- $\int (e^{3x} - e^{2x} + e^x) dx = \frac{1}{3} e^{3x} - \frac{1}{2} e^{2x} + e^x + c$
- $\int \frac{(y-2)^2}{\sqrt{y}} dy = \frac{2}{5} y^{5/2} - \frac{8}{3} y^{3/2} + 8y^{1/2} + c$

Rules Involving Substitution

Rule VI: The Substitution Rule

$$\int f(u)\frac{du}{dx}dx = \int f(u)du = F(u) + c$$

Rule VII: Integration by Parts**

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Let u = f(x) and v = g(x). Then,

$$\int v du = uv - \int u dv$$

Examples: Integration involving substitution

- Examples: Find integrals of

a. $\int xe^{x}dx$

Ans. $\int xe^{x}dx = e^{x}(x-1)+c$

b. $\int \frac{1}{x}\ln(x)dx$

Ans. $\int \frac{1}{x}\ln(x)dx = \frac{(\ln x)^{2}}{2}+c$

Initial-Value Theorem

- From $\int f(x)dx = F(x) + C$

If we have an initial condition, we can determine the value of C.

Example 1: Find F(x) if F'(x) = ½ - 2x and F(0) = ½.

Ans. $F(x) = \frac{1}{2}x - x^2 + \frac{1}{2}$

Example 2: Find F(x) if F'(x) = x(1-x^2) and F(1) = 5/12.

Ans. $F(x) = \frac{x^2}{2} - \frac{x^4}{4} + \frac{1}{6}$

Application 1: Derivation of TR from MR

- $TR = \int MR(Q)dQ$
- Example: Let MR = 10 – 2Q , what is the TR function?

Ans. $TR(Q) = 10Q - Q^2 = (10-Q)Q$

Application 2: Derivation of TC from MC

- $TC = \int MC(Q)dQ$
- Example: Let $MC = C'(Q) = 2e^{0.2Q}$.

  If the fixed cost is C_F = 90, what is the TC function?

  Ans. $TC(Q) = 10e^{0.2Q} + 80$

Application 3: Derivation of Profit Function from MR-MC

- π' = MP = MR – MC. ➔ $\pi = \int \pi'(Q)dQ$
- Example: Let MR = 50 – 2Q and MC = 10 + Q. Find total profit when Q = 10. Assume that there is no fixed cost.

Ans. $\pi(Q) = 40Q - 1.5Q^2$

Application 4: Derivation of Utility Function from MU

- $U(x) = \int MU(x)dx$
- Example: Let $MU(x) = \dfrac{5}{3\sqrt{x}}$. Find the utility function.

Ans. $U(x) = \dfrac{10}{3}\sqrt{x} + c$

Application 5: Derivation of Consumption/Saving Functions from Marginal Propensity Function

- Suppose the marginal propensity to save (MPS) function is:

  $S'(Y) = 0.3 - 0.1Y^{-1/2}$, and S = 0 when Y = \$81.

  Find the saving and consumption functions.

Ans. $S(Y) = 0.3Y - 0.2\sqrt{Y} - 22.5$

$C(Y) = 0.7Y + 0.2\sqrt{Y} + 22.5$

Definite Integrals

- $\int_a^b f(x)dx$ is a "definite" integral of f(x) from a to b (a < b), where
  - a = the lower limit of integration
  - b = the upper limit of integration

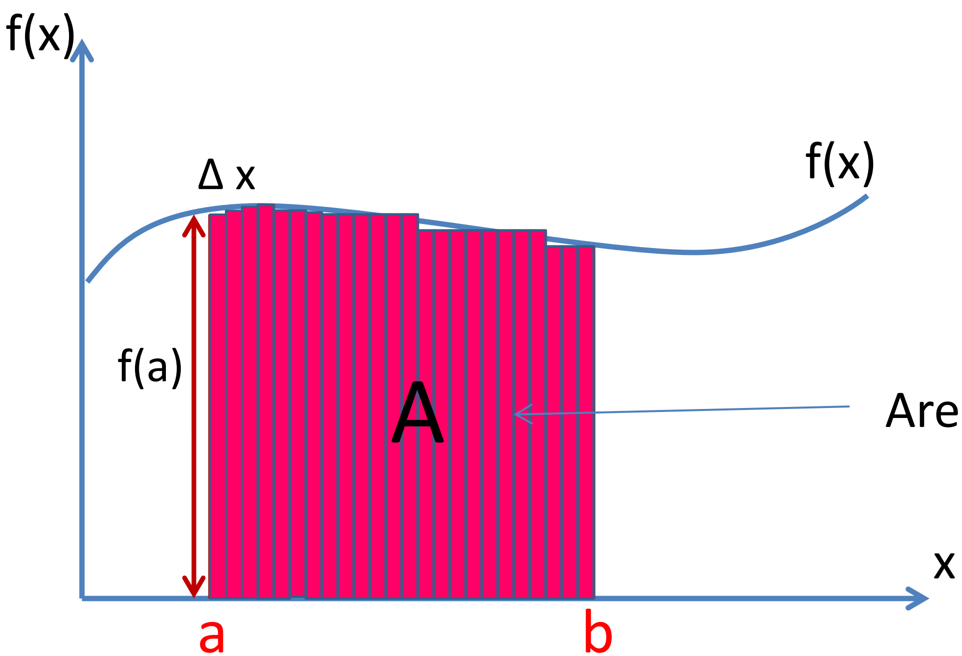
$$\Rightarrow \quad \boxed{\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)}$$

where F(x) = an arbitrary indefinite integral of f(x).

- Example: Find integral of
  - $\int_1^5 3x^2 dx$ = 124
  - $\int_0^1 \alpha e^{\beta\tau} d\tau$ = $\frac{\alpha}{\beta}\left(e^{\beta} - 1\right)$

Area and Definite Integrals

- The area under the graph of a continuous and nonnegative function f(x) over the interval [a, b] is $\int_a^b f(x)dx$.



Area A = $\lim_{\Delta x \to 0} \sum [f(x) \times \Delta x]$
$= \int f(x)dx$

Properties of Definite Integrals (I)

Property I: The interchange of the limits of integration changes of the sign of the definite integral:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Property II: A definite integral has a value of zero when the two limits of integrations are identical:

$$\int_a^a f(x)dx = 0$$

Property III: A definite integral can be expressed as a sum of a finite number of definite sub-integrals as follows:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, (a < c < b)$$

Properties of Definite Integrals (II)

Property IV:

$$\int_a^b - f(x)dx = -\int_a^b f(x)dx$$

Property V:

$$\int_a^b \alpha f(x)dx = \alpha\int_a^b f(x)dx$$

Property VI:

$$\int_a^b \left[f(x) + g(x)\right]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Property VII: (Integration by part)

$$\int_{x=a}^{x=b} v\,du = uv\Big|_{x=a}^{x=b} - \int_{x=a}^{x=b} u\,dv$$

Examples: Definite Integrals

Find the integrals of

- $\int_0^5 (x + x^2)dx = \frac{325}{6}$
- $\int_2^4 x^2 \left(\frac{1}{3}x^3 + 1\right) dx = \frac{1184}{9}$
- $\int_{-2}^2 (e^x - e^{-x})dx = \left(e^x - e^{-x}\right)\Big|_{-2}^{2} = 0$
- $\int_e^6 \left(\frac{1}{x} + \frac{1}{1+x}\right) dx = \left(\ln x + \ln|1+x|\right)\Big|_e^6 = (\ln 6 + \ln 7) - (1 + \ln(e+1))$
- $\int_{-2}^3 |x + 1| dx = \int_{-2}^{-1} -(x+1)dx + \int_{-1}^{3} (x+1)dx = 0.5 + 8 = 8.5$

Application 1: Capital Formation and Investment functions

- Definitions:
  - *K(t)* = capital stock at time t
  - *dK/dt* = the rate of capital formation
  - *I(t)* = the rate of net investment flow at time t
- Relationship between capital stock and net investment:

$$\frac{dK}{dt} = \dot{K} \equiv I(t) \quad \Rightarrow \quad \boxed{K(t) = \int I(t)dt = \int \frac{dK}{dt}dt = \int dK}$$

- Gross investment: $I_g(t) = I(t) + \delta K(t)$
- Capital formation during a time interval [a, b]:

$$\int_a^b I(t)dt = K(t)\Big]_a^b = K(b) - K(a)$$

Capital Formation and Investment functions (Cont'd)

- Example: Suppose the net investment flow is $I(t) = 3t^{1/2}$ and the initial capital stock at time t = 0 is $K(0) = 25$.

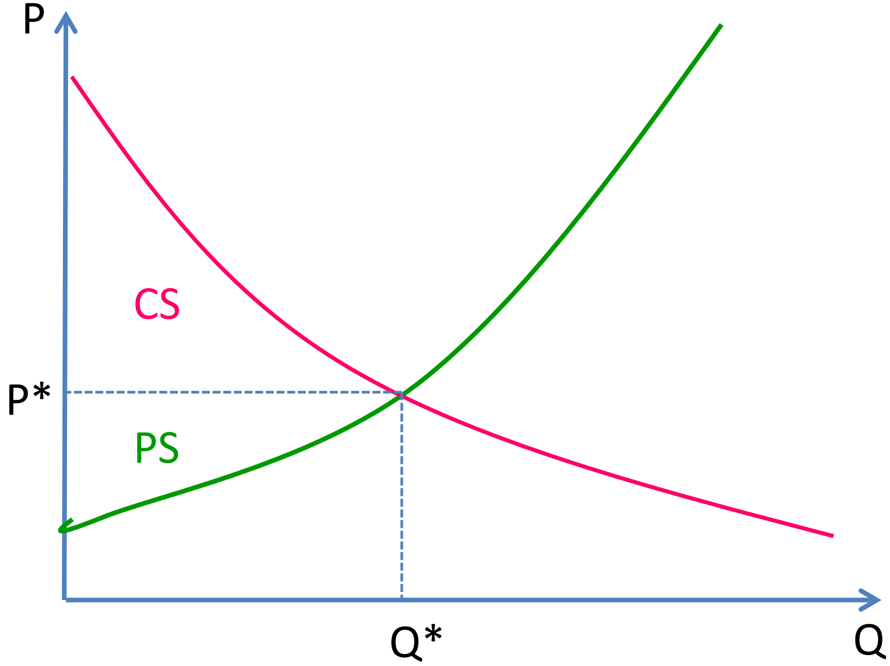
1. What is the time path of capital K?

Ans. $$K(t) = \int I(t)dt = \int 3t^{1/2}dt = 2t^{3/2} + 25$$

2. What is the capital formation during the time interval [1, 4]?

Ans. $$K(4) - K(1) = \int_1^4 I(t)dt = 14$$

Application 2: Consumer & Producer Surpluses



$$CS = \int_0^{Q^*} [D(Q) - P^*] dQ$$

$$PS = \int_0^{Q^*} [P^* - S(Q)] dQ$$

Example: Consumer Surplus & Producer Surplus

- Given a supply function $S(P) = -\frac{1}{2} + \frac{1}{2}P$ and a demand function $D(P) = \frac{25}{2} - \frac{1}{2}P$. Calculate: 1) producer and consumer surplus and 2) total welfare.

➢ Ans. P* = 13, Q* = 6

CS = 36; PS = 36

Welfare Effects of Price Change

- From the previous example, if the demand changes so that $D(P)=\frac{30}{2}-\frac{1}{2}P$, what happens to consumer and producer surplus?

➢ Ans. P* = 15.5, Q* = 7.25

CS = PS = 52.5625

Welfare Effects of Tax Distortion

- From the previous example, if the government imposes a $4 per unit tax on the producer, calculate the total welfare loss.

➢ Ans. $P_d^* = 15$, $P_s^* = 11$, $Q^* = 5$

CS = PS = 25 ➔ ΔCS = ΔPS = 25-36 = -11

Deadweight Loss = Total Welfare loss - Tax

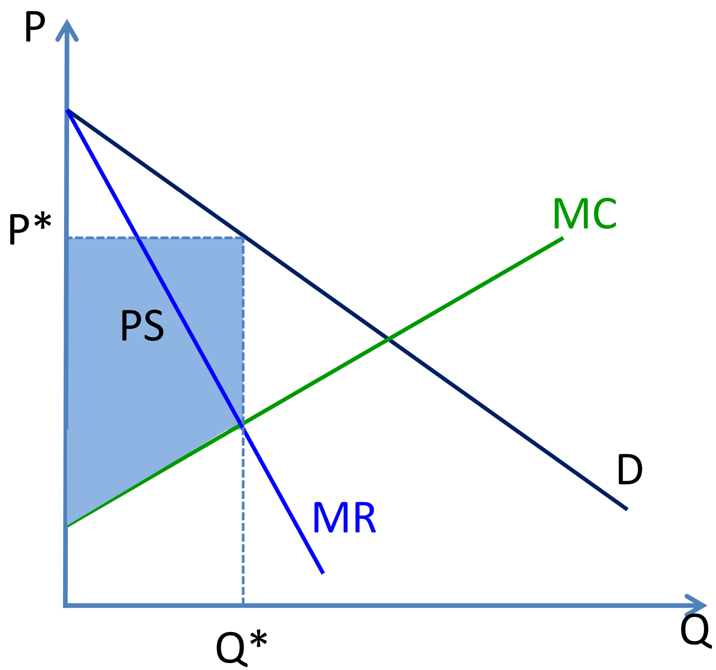
= (11+11) -(4*5)

= 2

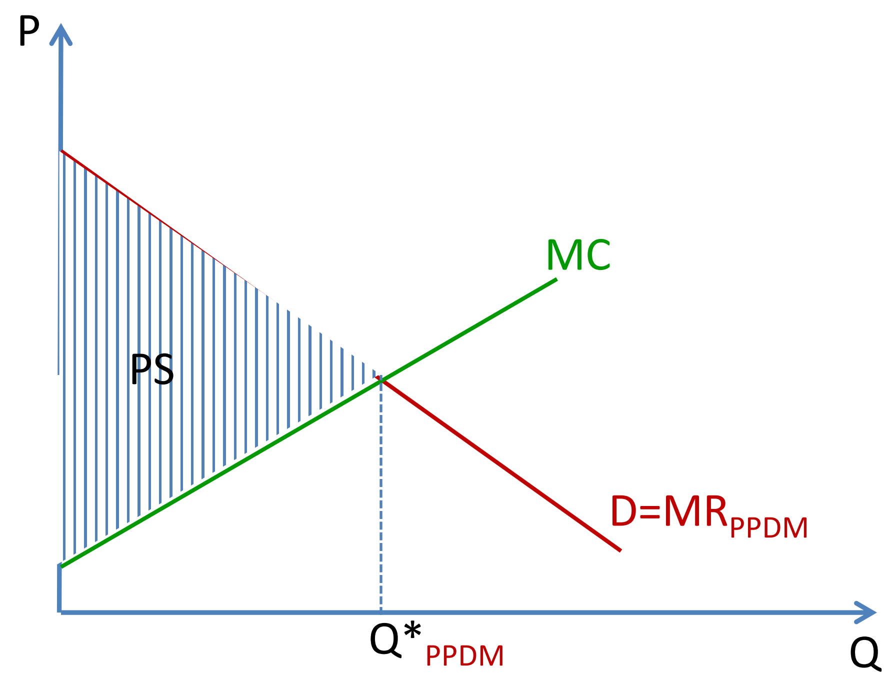
Application 3: First-Degree (or Perfect) Price Discrimination

- Perfect price discrimination occurs when the monopolist can charge the *maximum price for each unit of output sold*.

- (Regular) Monopolist



- Perfect price discriminating monopolist



Example: Perfect Price Discrimination (1)

- Suppose that a monopolist faces a demand function P = 24-Q and MC = 4 + 3Q. Find the consumer surplus at the profit-maximizing quantity. (No price discrimination)

➢ π-max: MR = MC ➔ 24 – 2Q = 4 + 3Q

➔ Q* = 4 ; P* = 20

➢CS = $\int_0^4 [(24 - Q) - 20]dQ = 8$

➢PS = $\int_0^4 [20 - (4 + 3Q)]dQ = 40$

Example: Perfect Price Discrimination (2)

- If the monopolist can practice perfect price discrimination, find the total revenue that maximizes the profit.

➢π-max: D = MC ➔ 24 –Q = 4 + 3Q

➔ Q* = 5

➢ PS = $\int_0^5 [(24 - Q) - (4 + 3Q)]dQ = 50$

➢TR = $\int_0^5 (24 - Q)dQ = 107.5$