Tutorial
Matrix Algebra

1. Find all solution to the following system of equations

(a)

$$\begin{aligned}3x+4y+z&=1\\2x+3y&=0\\4x+3y-z&=-2\end{aligned}$$

(Ans: x=-3/7, y=2/7,z=8/7)

(b)
$$\begin{aligned}x+2y-z&=2\\2x+5y+2z&=-1\\7x+17y+5z&=-1\end{aligned}$$

answer

$$\begin{aligned}x&=12+9z\\y&=-5-4z\end{aligned} \text{ (parametric form)}$$

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}12\\-5\\0\end{bmatrix}+C\begin{bmatrix}9\\-4\\1\end{bmatrix}$$; C∈R (vectors form)

c)
$$\begin{aligned}x+10z&=5\\3x+y-4z&=-1\\4x+y+6z&=1\end{aligned}$$

(ans: No solution)

2. Compute the rank of $A=\begin{bmatrix}1&1&-1&4\\2&1&3&0\\0&1&-5&8\end{bmatrix}$

(ans: 2)

3.Find a condition on the number a,b,c such that the following system of equations is consistent. When that condition is satisfied, find all solutions (in term of a,b, c)

$$\begin{aligned}x+3y+z&=a\\-x-2y+z&=b\\3x+7y\text{-}z&=c\end{aligned}$$

Ans: Consistent if c=a-2b

$$\begin{aligned} x &= 5z-(2a+3b) \\ y &= (a+b)-2z \quad \text{(Pararametric form)} \\ \text{Z} &= \text{free variables} \end{aligned}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -(2a+3b) \\ a+b \\ 0 \end{bmatrix} + C\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \quad \text{C}\in\text{R (vectors form)}$$

4. Suppose a square matrix A satisfies $A = 2A^T$. Show that necessarily A=0.

Solution

$$A = 2(2A^T)^T = 2(2(A^T)^T = 4A$$
$$3A = 0$$
$$A = 0$$

5. In each case find the matrix A

(a) $\left(3A^T + 2\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right)^T = \begin{bmatrix} 8 & 0 \\ 3 & 1 \end{bmatrix}$ Ans: $A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$

(b) $\left(2A^T - 5\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}\right)^T = 4A - 9\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$ Ans: $A = \begin{bmatrix} 2 & 7 \\ \frac{-9}{2} & -5 \end{bmatrix}$

6. Find the inverse of the matrix

$A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$ Ans: $A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -3 & 11 \\ 1 & 1 & -3 \\ 1 & -1 & -1 \end{bmatrix}$

7. Find A if $\left(A^T - 2I\right)^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$ Ans: $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

8. Suppose A is 3 by 5, B is 5 by 3, C is 5 by 1, and D is 3 by 1. Which of these matrix operations are allowed, and what are the shape of the results?

BA, A(B+C), ABD, AC+BD, ABABD

9. Compute AB where $A=\begin{bmatrix} 2 & 0 & -3 \\ -1 & 6 & 5 \end{bmatrix}$ and $B=\begin{bmatrix} 0 & 3 \\ -4 & 7 \\ 2 & -5 \end{bmatrix}$. Show how you obtain the result using all three methods you study in the class.

10. Find all solutions to the following system of linear equations:

$$\begin{aligned} x+y-3z &= 3 \\ -2x-y &= -4 \\ 4x+2y+3z &= 7 \end{aligned}$$

11. Determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system

a) $\left[\begin{array}{cc|c} 2 & 3 & h \\ 4 & 6 & 7 \end{array}\right]$ b) $\left[\begin{array}{cc|c} 1 & -3 & 2 \\ 5 & h & -7 \end{array}\right]$

12. For matrix

$$\mathbf{A}=\begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 2 & 8 & 0 \end{bmatrix}$$

determine the echelon form U, the basic variables, the free variables and the general solution to Ax=0. Then apply, elimination to Ax=b , with b_1 and b_2 on the right side; find the conditions for Ax=b to be consistent and find the general solution. What is the rank of A.

13. a) Find all solutions to

$$\mathbf{Ux}=\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

b) if the right side is changed from (0,0,0) to (a,b,0), what are the solutions?

14. Use row operation to find

$$\begin{aligned} 6x_1 + x_2 + x_3 &= 6 \\ 5x_1 + x_2 + 2x_3 &= 4 \\ 4x_1 + x_2 - x_3 &= -2 \end{aligned}$$

15. Suppose

$$Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \text{has no solution}$$

but

$$Ax = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \text{ has infinitely many solutions.}$$

a) Find all possible information about r, m and n. (r is the rank and A is of the size m×n)

b) Find an example of such a matrix A with r, m and n all as small as possible.

16. a) By elimination put A into its upper triangular form U. Which are the pivot columns and free columns?

$$A = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 2 & 8 & 5 & 2 \\ 1 & 5 & 3 & 1 \end{bmatrix}$$

b) Describe specifically the vectors which are solutions to Ax=0

c) Does Ax=b have a solution for the right side b $= \begin{bmatrix} 3 \\ 8 \\ 5 \end{bmatrix}$? If it does, find one particular solution and then complete solution to this system Ax=b.

17. a) Show that $A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$ has inverse $C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

b) Show that $A = \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}$ has no inverse.

18. a) Find the inverse of the matrix $A = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 7 & 1 \\ 1 & 3 & 0 \end{bmatrix}$ using Gauss-Jordan method.

c) show that the matrix $A = \begin{bmatrix} 1 & 3 & -2 \\ 1 & 2 & 0 \\ 2 & 8 & -8 \end{bmatrix}$ is not invertible.

19. Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

(a) Use Gauss-Jordan elimination to compute the inverse of the matrix A.

(b) Use the answer in part (a) to solve the system $Ax = b$.

20. Compute the determinant of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 0 & 0 \\ 0 & 7 & 8 & 0 \\ 0 & 0 & 9 & 10 \end{bmatrix}.$$

21. Suppose the 4 by 4 matrix M has four equal rows all containing a,b,c,d. We know that det (M) = 0. The problem is to find by any method

$$\det(I+M) = \begin{vmatrix} 1+a & b & c & d \\ a & 1+b & c & d \\ a & b & 1+c & d \\ a & b & c & 1+d \end{vmatrix}$$