Curve Sketching

1 Curve Sketching

Let f be a relation from X to Y where $X, Y \subseteq \mathbb{R}$.
Recall that we can write f as

$$f = \{(x, y)|x \in X, y \in Y\}$$

or we can write $y = f(x)$, where $x \in X$. We are interested in how to sketch a graph or curve of a give relation or function f.

Definition 1.1 (Graph/Curve)**.** Let X and Y be subsets of $\mathbb{R}$. For a relation $f : X \to Y$, each ordered pair $(x, f(x))$, $x \in X$, can be represented by a point in the Cartesian plane. The collection of all such points is called the **graph or curve of** f.

Curve Sketching
Let f be a relation from X to Y where $X, Y \subseteq \mathbb{R}$. To sketch a curve for f, it is helpful to identify the following characteristics of the curve.

(1) Domain and Range (undefined points)

(2) Intercepts: the x-and y-intercepts

(3) Symmetry: about the x-axis/y-axis/origin

(4) Asymptotes: Vertical/Horizontal/Slant Asymptotes

(5) Increasing/Decreasing Intervals

(6) Local (Relative) Maximum/Minimum

(7) Concavity and Points of Inflection

By using the information obtained from steps (1) to (7)., we can sketch the curve as follows.

- First, draw dashed lines for the asymptotes of the function.
- Then plot the x-and y-intercepts, maximum and minimum points and points of inflection on the graph.
- Sketch the curve between the points, using the intervals of increase and decrease and intervals of concavity.
- Be sure that the graph behaves correctly when approaching asymptotes.

2 Curve Sketching I

2.1 Domain & Range, Intercepts, Symmetry

Remarks

- The steps (1)-(3) can be done by using some basic algebra knowledge.
- For steps (4)-(7), the basic calculus knowledge would be very helpful to identify these characteristics.

Curve Sketching for $f : X \to Y$, $X, Y \subseteq \mathbb{R}$

(1) **Domain & Range** of f: Find the set of all values x such that f is well-defined and the corresponding y. This will be useful when finding vertical asymptotes and determining critical numbers.

(2) **Intercepts:** Find the x-and y-intercepts for $(x, y) \in f$, if possible.

- To find the x-intercept, set $y = 0$ and solve the equation for x.
- To find the y-intercept, set $x = 0$ and solve the equation for y.

(3) **Symmetry:** The curve is...

- **Symmetric about the x-axis** if $\boxed{(x, y) \in f \text{ implies } (x, -y) \in f.}$
- **Symmetric about the y-axis** if $\boxed{(x, y) \in f \text{ implies } (-x, y) \in f.}$
  When f is a function, denoted by $y = f(x)$, the condition above becomes
  $$\boxed{f(x) = f(-x), \quad \forall x \in X}$$
  and the function f is called *"even."*
- **Symmetric about the origin** if $\boxed{(x, y) \in f \text{ implies } (-x, -y) \in f.}$
  When f is a function, denoted by $y = f(x)$, the condition above becomes
  $$\boxed{f(-x) = -f(x), \quad \forall x \in X}$$
  and the function f is called *"odd."*

Example 2.1. Determine the symmetry of the graph for each of the following relations.

1. $f_1 = \{(x, y) | y = x^2\}$

2. $f_2 = \{(x, y) | x = y^2\}$

3. $f_3 = \{(x, y) | y = -x\}$

4. $f_4 = \{(x, y) | x^2 + y^2 = 4\}$

3 Curve Sketching II

3.1 Asymptotes

(4) Asymptotes of a curve can be determined in many different ways. The followings are common two approaches: (i) re-arranging the term and (ii) using the limit.

(i) Re-arranging the term: Consider the relation of the form

$$F(x, y) = C$$

such that $F(x, y))$ can be factored into linear terms, *e.g.*$x - a, y - b, mx + ny$, and C is a nonzero constant.

- **The vertical asymptote**:line $x = a$
- **The horizontal asymptote**: line $y = b$
- **The slant asymptote**: line $mx + ny = 0$

E.g.

- To obtain a **vertical asymptote** (e.g. line $x = a$), arrange the equation in the form $y = f(x)$ and find a such that the *denominator* of $f(a)$ is *zero*.
  Note: In this case, a is not in the domain of f.

- To obtain a **horizontal asymptote** (e.g. line $y = b$), arrange the equation in the form $x = g(y)$ and find b such that the *denominator* of $g(b)$ is *zero*.
  Note: In this case, b is not in the range of f.

(ii) Using the limit: This method is often used when the relation f is a function $f : X \to Y$, $y = f(x), x \in X$.

- **Vertical asymptote**: By using the equation in the form $y = f(x)$,
  $\boxed{x = a}$ is a vertical asymptote if

$$\boxed{\begin{array}{l} \lim_{x\to a} f(x) = \infty \quad \text{or} \quad \lim_{x\to a^+} f(x) = \infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \infty \quad \text{or} \\ \lim_{x\to a} f(x) = -\infty \quad \text{or} \quad \lim_{x\to a^+} f(x) = -\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = -\infty. \end{array}}$$

  Alternatively, if it is possible to re-arrange $x = g(y)$, $x = a$ is a vertical asymptote when $\lim_{y\to\infty} g(y) = a$ or $\lim_{y\to-\infty} g(y) = a$.

- **Horizontal asymptote**: By using the equation in the form $y = f(x)$,
  $\boxed{y = b}$ is a horizontal asymptote if

$$\boxed{\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.}$$

  Alternatively, if it is possible to re-arrange $x = g(y)$, $y = b$ is a horizontal asymptote when $\lim_{y\to b} g(y) = \infty$ or $\lim_{y\to b^+} g(y) = \infty$ or $\lim_{y\to b^-} g(y) = \infty$ or $\lim_{y\to b} g(y) = -\infty$ or $\lim_{y\to b^+} g(y) = -\infty$ or $\lim_{y\to b^-} g(y) = -\infty$.

- **Slant asymptote**: By using the equation in the form $y = f(x)$, the line $\boxed{y = mx + b}$ is a slant asymptote if

$$\boxed{\lim_{x\to\infty}[f(x) - (mx + b)] = 0 \quad \text{or} \quad \lim_{\text{x}\to-\infty}[\text{f(x)} - (\text{mx} + \text{b})] = 0.}$$

Slant asymptotes for rational functions: A slant (oblique) asymptote occurs when the polynomial in the numerator is a higher degree than the polynomial in the denominator by 1.
To find the slant asymptote you must divide the numerator by the denominator using either long division or synthetic division.

The following procedure can be used for finding the asymptotes for **rational functions**.

- Set the denominator equal to zero and solve for the roots. These zeroes/roots (if any) give the **vertical** asymptotes (everything else is in the domain).
- compare the degrees of the numerator and the denominator
  1. If the degrees are the same, then you have a **horizontal** asymptote at:
     y = (numerator's leading coefficient) / (denominator's leading coefficient).
  2. If the denominator's degree is greater (by any margin), then you have a horizontal asymptote at y = 0 (the x-axis).
  3. If the numerator's degree is greater (by a margin of 1), then you have a slant asymptote which you will find by doing long division.

Example 3.1. Determine the slant asymptote of the function $f(x) = \frac{3x^2+x-2}{2x+6}$ (if any).

Example 3.2. Consider the relation defined by

$$f = \{(x, y)|x^2 + xy - x - y - 3 = 0\}.$$

Find all asymptotes for the curve of f.

Example 3.3. Consider a function

$$f(x) = \frac{x^2 + 5x + 4}{x^2}.$$

1. Find domain and range of f.
2. Find x-intercepts and y-intercepts.
3. Determine the symmetry of f.
4. Find the asymptotes for f (if any).

The characteristics (5)-(7), increasing/decreasing, extrema, and concavity, of the curves of a relation $f : X \to Y$ can generally be identified when f is a *continuous* and *differentiable* function on a subset of its domain $x \subseteq \mathbb{R}$ by using some basic *calculus*. The followings are some calculus required to used for identifying these characteristics.

Definition 3.1. A critical number of a function $f : X \to Y$ is a number c in its domain X for which

$$f'(c) = 0, \qquad \text{or} \qquad f'(c) \quad \text{does not exist.}$$

Example 3.4. Find the domain and the critical numbers for each of the following functions.

- $f(x) = (x+4)^{2/3}$
- $f(x) = \frac{x^2}{x-1}$.

3.2 Increasing/Decreasing

(5) **Increasing/Decreasing Intervals:** (defined below) can be identified by using the first derivative from the following theorem.

Definition 3.2 (Increasing/Decreasing Function). Let $f : X \to Y$ be a function $y = f(x)$, $X, Y \subseteq \mathbb{R}$. Let $S \subseteq X$.

- f is increasing on S if and only if, for all $x_1, x_2 \in S$, if $x_1 < x_2$, then $f(x_1) < f(x_2)$.
- f is decreasing on S if and only if, for all $x_1, x_2 \in S$, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.

Theorem 3.1. Test for Increasing/Decreasing Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b).

- If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.
- If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.
- If $f'(x) = 0$ for all x in $[a, b]$, then f is a constant on the interval.

3.3 Extrema

(6) **Extrema:** (defined below) of f can be identified by using the first derivatives and the second derivatives from the following theorems.

Let $f : X \to Y$, $X, Y \subset \mathbb{R}$ be a function.

Definition 3.3 (Relative Extrema: Maximum/Minimum).

- A number $f(c)$ is a **relative maximum** of a function f if $f(x) \leq f(c)$ for every x in *some* open interval that contains c.
- A number $f(c)$ is a **relative minimum** of a function f if $f(x) \geq f(c)$ for every x in *some* open interval that contains c.

Theorem 3.2. Relative Extrema Occur at Critical numbers If a function f has a relative extremum at $x = c$, then c is a critical number.

From the previous theorem, we can find an extremum by first find all critical numbers c of f first. Then, we can use two following approaches to identify if each critical number gives maximum, minimum, or it does not give any extremum.

- The First Derivative Test for Relative Extrema
- The Second Derivative Test for Relative Extrema

Theorem 3.3. First Derivative Test for Relative Extrema Let f be continuous on $[a, b]$ and differentiable on (a, b) except possibly at the critical number c.

(i) If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a relative **maximum**.

(ii) If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a relative **minimum**.

(iii) If $f'(x)$ has the same algebraic sign on each side of c, then $f(c)$ is **not an extremum**.

Theorem 3.4. Theorem: Second Derivative Test for Relative Extrema Let f be a function for which the second derivative f'' exists on an interval (a, b) that contains the critical number c.

(i) If $f''(c) > 0$, then $f(c)$ is a relative **minimum**.

(ii) If $f''(c) < 0$, then $f(c)$ is a relative **maximum**.

(iii) If $f''(c) = 0$, the test fails and $f(c)$ may or may not be a relative extremum. In this case, use the First derivative Test.

Notice that, to find an extremum, we have to first find all critical numbers c of f.

Example 3.5. Consider the function $f(x) = x^3 - 3x^2 - 9x + 2$

1. Find domain of f.

2. Find x-intercepts and y-intercepts (if any).

3. Determine the symmetry of f. Determine whether f is even or odd.

4. Find the asymptotes for f (if any).

5. Find the critical number of f.

6. Determine the intervals on which f is increasing and decreasing.

7. Determine the extrema (maximum and minimum) of f by using
   - the First Derivative Test,
   - the Second Derivative test.

3.4 Concavity & Inflection Point

(7) **Concavity and the point of inflection:** (defined below) of f can be identified by using the the second derivatives from the following theorem.

Let $f : X \to Y$ be a function with $X, Y \subseteq \mathbb{R}$.

Definition 3.4 (Concavity: concave up/concave down)**.** Let f be a differentiable function on an interval (a, b).

(i) If f' is an increasing function on (a, b), then the graph of f is **concave up** on the interval.

(ii) If f' is an decreasing function on (a, b), then the graph of f is **concave down** on the interval.

Definition 3.5 (Point of Inflection)**.** Let f be a continuous on an interval (a, b) containing the number c. A point $(c, f(c))$ is a **point of inflection** of the graph f if there is a tangent line at $(c, f(c))$ and the graph changes concavity at this point.

Theorem 3.5. Test for Concavity Let f be function for which f'' exists on (a, b).

(i) If $f''(x) > 0$ for all x in (a, b), then the graph of f is concave up on (a, b).

(ii) If $f''(x) < 0$ for all x in (a, b), then the graph of f is concave down on (a, b).

Theorem 3.6. Point of Inflection
If $(c, f(c))$ is a point of inflection for the graph of a function f, then either

$$f''(c) = 0 \qquad \text{or} \qquad f''(c) \text{ does not exist.}$$

Example 3.6. Consider the function $f(x) = 4x^4 - 4x^2$.

1. Find domain of f.
2. Find x-intercepts and y-intercepts (if any).
3. Determine the symmetry of f. Determine whether f is even or odd.
4. Find the asymptotes for f (if any).
5. Find the critical numbers of f.
6. Determine the intervals on which f is increasing and decreasing.
7. Determine the relative extrema (maximum and minimum) of f
8. Use the information above to sketch the graph of f.

Example (continued)