Note Title 9/2/2014

Last time 2.1 First-Order linear DE

2.1.1 constant coefficient and constant term

$$\dot{y}(t) + ay(t) = b \quad \text{——} \quad (4)$$

Homogeneous case $b = 0 \rightarrow y(t) = Ae^{-at}$ Gen Sol$^{\underline{n}}$

$y(t) = y(0)e^{-at}$ Def Sol$^{\underline{n}}$

NonHomogeous case $a, b \neq 0$

$$y(t) = y_c + y_p$$

To get y_c, we set $b = 0$ (Homo case) $\Rightarrow y_c = Ae^{-at}$

To get y_p, we set $y_p = k \rightarrow \frac{dy}{dt} = 0$

sub into the eqn (4)

$$0 + ak = b$$

$$k = \frac{b}{a}$$

$$\therefore \quad y_p = k = \frac{b}{a} \qquad a \neq 0$$

$$\therefore \quad \boxed{\begin{aligned} y_t &= y_c + y_p \\ &= Ae^{-at} + \frac{b}{a} \\ y_t &= \left[y(0) - \frac{b}{a}\right]e^{-at} + \frac{b}{a} \end{aligned}}$$

$\Rightarrow$ Gen Sol$^{\underline{n}}$

If $a \neq 0$

$\Rightarrow$ Def Sol$^{\underline{n}}$

What if $a = 0$?

and we set $y_p = k \rightarrow y'(t) = 0$

sub into (4)

$$0 + \overset{=0}{a} \cdot k = b$$

$$0 = b$$

$\therefore$ when $a = 0$, the guess $y_p = k$ doesn't work

so we try $y_p = kt \rightarrow \frac{dy}{dt} = k$

sub into (4) $\quad k + \underset{=0}{a}\underset{=y_p}{(kt)} = b$

$$k + 0 = b$$

$$k = b$$

$$y(t) = y_c + y_p$$

$$= y_c + kt$$

$$= y_c + bt \qquad ; k = b, \; a = 0$$

$y(t) = A + bt \qquad ; a = 0 \qquad$ Gen Sol^n

$y(t) = y(0) + bt \qquad ; a = 0 \qquad$ Def Sol^n

2.1.2 Dynamics of Mkt Price

$$Q_d = \alpha - \beta P \qquad (\alpha, \beta > 0)$$

$$Q_s = -\gamma + \delta P \qquad (\gamma, \delta > 0)$$

$$P^* = \frac{\alpha + \gamma}{\beta + \delta} \qquad \text{— (10)}$$

The time path of price

$$\frac{dP}{dt} = j\,(Q_d - Q_s) \quad ; \; j > 0 \qquad \text{— (11)}$$

j = a (constant) adjustment coefficient.

$$\frac{dP}{dt} = 0 \quad \Leftrightarrow \quad Q_d = Q_s$$

<u>Equilibrium Price</u>

1) The mkt-clearing sense $\Rightarrow$ P that equates Q_d and Q_s

2) The intertemporal sense ; P is constant over time

$P^* = P(0)$

sub Q_d, Q_s from (9) $\rightarrow$ (11)

$$\frac{dP}{dt} = j\left[\overbrace{(\alpha - \beta P)}^{Q_d} - \overbrace{(-\gamma + \delta P)}^{Q_s}\right]$$

$$\frac{dP}{dt} = j(\alpha + \gamma) - j(\beta + \delta)P$$

$$\frac{dP}{dt} + j(\beta + \delta)P = j(\alpha + \gamma)$$

$$\frac{dy}{dt} + ay = b \qquad y(t) = \left[y(0) - \frac{b}{a}\right]Ae^{-at} + \frac{b}{a}$$

$$P(t) = \left[P(0) - \frac{\cancel{j}}{\cancel{j}}\left(\frac{\alpha + \gamma}{\beta + \delta}\right)\right]e^{-j(\beta+\delta)t} + \frac{\cancel{j}}{\cancel{j}}\left(\frac{\alpha + \gamma}{\beta + \delta}\right)$$

$$P(t) = [P(0) - P^*]e^{-kt} + P^* \quad ; \; k = j(\beta + \delta)$$

— (11)'

Since $P(0)$ and P^* $\Rightarrow$ constant

The key factor 'e^{-kt}' determines the time path of price

since $k \geq 0$, $t \rightarrow \infty \Rightarrow e^{-kt} \rightarrow 0$

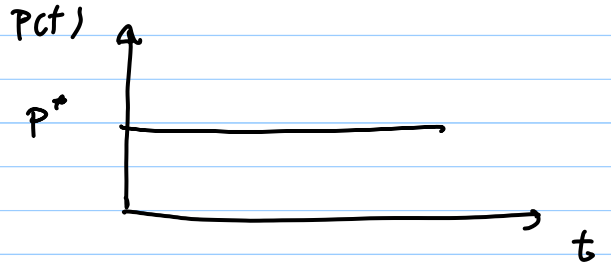
The time path will lead the price toward the equilibrium point. P^*

$P(t)$ converges to the level P^*

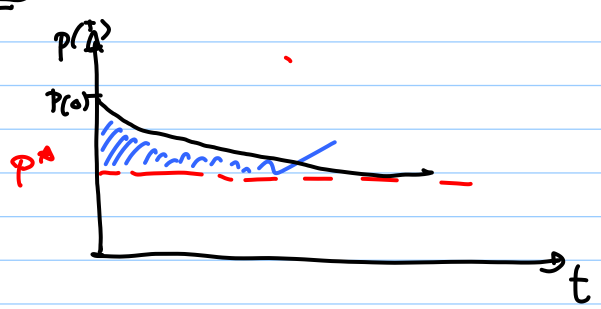
P^* is "Dynamically stable"

Case 1 : $P(0) \doteq P^*$

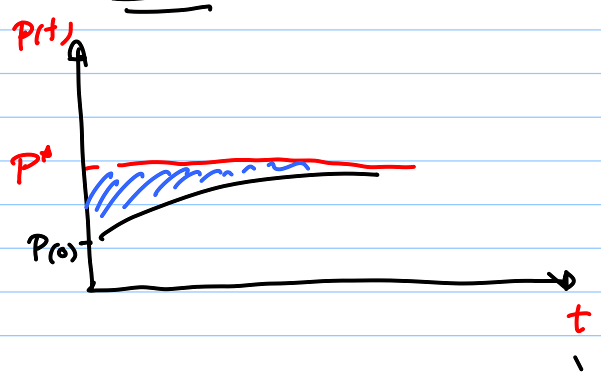
$\rightarrow P(T) = P^*$; $P(0) - P^* = 0$



Case 2 $P(0) > P^*$



Case 3 $P(0) < P^*$



$$y(t) = y_c + y_p$$

$$= \left[y(0) - \frac{b}{a} \right] e^{-at} + \frac{b}{a}$$

y_p = the intertemporal equilibrium level of $y(t)$

y_c = the deviation from equilibrium

NOTE If $P(0) \neq P^*$ $y_c \rightarrow 0$ as $t \rightarrow \infty$

iff $k > 0$

$j(\beta + \delta) > 0$

$Q_d = \alpha - \beta P$

$Q_s = -\gamma + \delta P$

since $j > 0$, $\beta + \delta > 0$

$\delta > -\beta$

2.1.3 Variable coefficient and Variable Term

$u(t), w(t)$ = continuous function of time t

$$\frac{dy(t)}{dt} + u(t)\,y(t) = w(t) \qquad \text{--- (12)}$$

@Homogeneous case

$$w(t) = 0 \quad \rightarrow \quad \frac{dy}{dt} + u(t)\,y = 0 \qquad \text{--- (13)}$$

$$\frac{1}{y}\cdot\frac{dy}{dt} = -u(t)$$

$$\int \frac{1}{y}\frac{dy}{dt}\,dt = -\int u(t)\,dt$$

$$\ln|y| + c = -\int u(t)\,dt$$

$$\ln|y| = -c - \int u(t)\,dt$$

$$e^{\ln|y|} = e^{-c}\cdot e^{-\int u(t)dt}$$

$$y(t) = A.e^{-\int u(t)dt} \quad ; \; A = e^{-c} \qquad \text{--- (14)}$$

(14) is general sol^n of (13)

Ex $$\dot{y}(t) + 3t^2 y = 0$$

$$u(t) = 3t^2 \Rightarrow \int u(t)\,dt = \int 3t^2\,dt$$
$$= t^3 + c$$

$$y(t) = A\,e^{-\int u(t)\,dt}$$

$$= A\,e^{-(t^3+c)} = A.e^{-c}\cdot e^{-t^3}$$

$$\boxed{y(t) = Be^{-t^3}} \qquad B = Ae^{-c}$$

Check $y'(t) = ?$

$\hookrightarrow \dot{y}(t) + 3t^2 y = 0$

ⓒ The Non Homogeneous Case

$$u(t), w(t) \neq 0$$

General Sol^n

$$\boxed{y(t) = e^{-\int u(t)\,dt}\left(A + \int w(t)\, e^{\int u(t)\,dt}\, dt\right)} \quad -(15)$$

Proof

$$\dot{y}(t) + u(t).y(t) = w(t)$$

multiplying by a chosen "integrating factor" $\Rightarrow e^{c(t)}$

$$\underline{\dot{y}e^{c(t)} + u(t)\,y\,e^{c(t)}} = w(t)e^{c(t)}$$

Note that

$$y.e^{c(t)} \rightarrow \text{its derivative w.r.t. } t \rightarrow \underline{\dot{y}e^{c(t)} + y\,\dot{c}(t)\,e^{c(t)}}$$

We thus make $C(t)$ satisfy $\dot{c}(t) = u(t)$

by choosing $c(t) = \int u(t)\,dt$

$$\therefore \quad \frac{d\, y e^{c(t)}}{dt} = w(t)e^{c(t)}$$

$$y.e^{c(t)} = A + \int w(t)e^{c(t)}\,dt \quad ; \; A = \text{constant}$$

$$y(t) = Ae^{-c(t)} + e^{-c(t)}\int w(t)e^{c(t)}dt$$

$$y(t) = e^{-c(t)}\left(A + \int w(t)e^{c(t)}dt\right)$$

since $c(t) = \int u(t)\,dt$

$$\boxed{y(t) = e^{-\int u(t)\,dt}\left(A + \int w(t)\, e^{\int u(t)\,dt}\,dt\right)}$$

21.4 Exact Differential Equations

Ex $\quad 2ty\dfrac{dy}{dt} + y^2 = 0$

Let $f(y,t) = y^2 \rightarrow \dfrac{\partial f}{\partial y} = 2y$

EXACT

$g(y,t) = 2ty \rightarrow \dfrac{\partial g}{\partial t} = 2y$

Ex $\quad \underbrace{1 + ty^2}_{} + t^2 y\,\dot{y} = 0$

$g(y,t) = t^2 y$

$f(y,t) = 1 + ty^2$

$\therefore \dfrac{\partial f}{\partial y} = 2ty$ - - - EXACT - - - $\therefore \dfrac{\partial g}{\partial t} = 2ty$

Now

Consider $\quad g(y,t)\dfrac{dy}{dt} + f(y,t) = 0 \qquad$ — (16)

Suppose we have a fn $F(y,t)$ such that

$$\left.\begin{aligned} \frac{\partial F(y,t)}{\partial t} &= f(y,t) \\ \text{and} \quad \frac{\partial F(y,t)}{\partial y} &= g(y,t) \end{aligned}\right\} \quad \text{— (17)}$$

This implies the total derivatives of $F(y,t)$

$$dF(y,t) = \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial t}dt$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial t}$$

$$\frac{dF}{dt} = \underbrace{g(y,t)\,\dot{y}(t) + f(y,t)}_{\text{LHS of (16)}}$$

$$\therefore \quad \frac{dF(y,t)}{dt} = 0$$

only iff $\quad F(y,t) = \text{constant } C$

The general sol^n should be in the form

$$\boxed{F(y,t) = C}$$