

## Chapter 3 Mathematical Induction

Carl Friedrich Gauss is considered to be one of the most brilliant mathematicians of all time. The story goes that when he was very young (in grade school in Germany) his teacher gave him and his classmates the supposedly unpleasant task of adding the integers from 1 to 100. He obtained the correct result of 5050 quickly. It is believed that he considered both the sum  $1+2+\dots+100$  and its reverse  $100+99+\dots+1$  and added these to obtain the sum  $101+101+\dots+101$ , which has 100 terms and so equals 10100. Since this is twice the required sum,  $1+2+\dots+100=10100/2=5050$ . This, of course, can be quite easily generalized to find a formula for  $1+2+\dots+n$ , where  $n \in \mathbb{N}$ . Let

$$S = 1 + 2 + \dots + n. \quad (3.1)$$

If we reverse the order of the terms on the right side of (3.1), then we obtain

$$S = n + (n-1) + (n-2) + \dots + 1 \quad (3.2)$$

Adding (3.1) and (3.2), we have

$$2S = (n+1) + (n+1) + \dots + (n+1) \quad (3.3)$$

Since there are  $n$  terms on the right side of (3.3), we conclude that  $2S = n(n+1)$  or  $S = n(n+1)/2$ . Hence

$$1 + 2 + \dots + n = \frac{n(n+1)}{2}.$$

This formula can be verified by a totally different technique. As we shall see, this technique called *mathematical induction*, or simply *induction*, is very powerful. First, we present a theorem, called the *Principle of Mathematical Induction*.

Suppose that we have a sequence of statements which we would like to show are true, say one statement for each positive integer. More specifically, let  $S_1, S_2, S_3, \dots$  be statements. We wish to show that all of these statements are true. Of course, it is not at all practical to establish the truth of these statements individually. An alternative is now presented.

**Theorem 3.1** (The Principle of Mathematical Induction) *Let  $S_1, S_2, S_3, \dots$  be statements satisfying the following properties:*

(1)  $S_1$  is true;

(2) if  $S_k$  is true, where  $k \in N$ , then  $S_{k+1}$  is true. (i.e.  $\forall k (S_k \rightarrow S_{k+1})$  where  $k \in N$ )

*Then all of the statements  $S_1, S_2, S_3, \dots$  are true.*

**Example 1:** Use mathematical induction to prove that

$$S_n : 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

is true for every positive integer  $n$ .

**Example 2:** Prove that for every positive integer  $n$ ,

$$\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(n+1)(n+2)} = \frac{n}{2n+4}$$

**Example 3:** Prove that for every positive integer  $n$ , 3 divides  $n^3 + 5n$ .

**Example 4:** Show that for every natural number  $n$ ,  $n < 2^n$ .

By a nearly identical argument, we can prove a somewhat more general result.

**Theorem 3.2** Let  $S_m, S_{m+1}, S_{m+2}, \dots$  be statements satisfying the following properties:

(1)  $S_m$  is true;

(2) if  $S_k$  is true, where  $k \in I$  and  $k \geq m$ , then  $S_{k+1}$  is true.

Then all of the statements  $S_m, S_{m+1}, S_{m+2}, \dots$  are true.

**Example 5:** Let  $n$  be positive integer and  $n \geq 2$ , prove that  $1 + 2n < 3^n$ .

**Example 6:** Let  $n$  be positive integer that is greater than 4, prove that  $n^2 + 1 < 2^n$ .

**Example 7:** Let  $n$  be positive integer that is greater than 3, prove that  $2^n < n!$ .