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6.5 Exercises

1. Two individuals agree at date 0 to a forward contract that matures at date 2. The contract is written on an underlying asset that pays a dividend at date 1 equal to D_1. Let f_2 be the date 2 random payoff (profit) to the individual who is the long party in the forward contract. Also let m_{0i} be the stochastic discount factor over the period from dates 0 to i where $i = 1, 2$, and let $E_0[\cdot]$ be the expectations operator at date 0. What is the value of $E_0[m_{02} f_2]$? Explain your answer.

- Let S_i be the price of the underlying asset at date i

$$
\begin{aligned}
P_{it} &= E_t \left[\sum_{j=1}^{\infty} \delta^j \frac{u_C(C^*_{t+j})}{u_C(C^*_t)} d_{i,t+j} \right] \\
&= E_t \left[\sum_{j=1}^{\infty} m_{t,t+j} d_{i,t+j} \right] \qquad (6.16)
\end{aligned}
$$

From equation (6.16), we can write

payoff at date 1

payoff at date 2

$$S_0 = E_0 \left[m_{01} D_1 + m_{02} S_2 \right]$$

$$= D_0 + E_0[m_{02} S_2] \quad ; \text{ where } D_0 = E_0[m_{01} D_1]$$

- Let F_{02} be the forward price, then we can write

$$f_2 = S_2 - F_{02} .$$

Thus, $E_0[m_{02} f_2] = E_0[m_{02}(S_2 - F_{02})] = E_0[m_{02} S_2] - E_0[m_{02} F_{02}]$

$$= (S_0 - D_0) - E_0[m_{02} F_{02}]$$

It is known at date 0.

Moreover, we know that $E_0[m_{02}F_{02}] = E_0[m_{02}]F_{02} = R_f^{-2}F_{02}$.

Hence, $$E_0[m_{02}f_2] = S_0 - D_0 - R_f^{-2}F_{02}$$

Along with no-arbitrage condition, $F_{02} = R_f^2(S_0 - D_0)$, we have

$$E_0[m_{02}f_2] = S_0 - D_0 - R_f^{-2}(R_f^2(S_0 - D_0)) = 0. \quad \#$$

2. Assume that there is an economy populated by infinitely-lived representative individuals who maximize the lifetime utility function

$$E_0\left[\sum_{t=0}^{\infty} -\delta^t e^{-ac_t}\right]$$

where c_t is consumption at date t and $a > 0$, $0 < \delta < 1$. The economy is a Lucas (1978) endowment economy having multiple risky assets paying date t dividends that total d_t per capita. Write down an expression for the equilibrium per capita price of the market portfolio in terms of the assets' future dividends.

From the lecture note, a result of Lucus (1978) is

$$P_0 = E_0\left[\sum_{t=1}^{\infty} \frac{U_c(C_t^*)}{U_c(C_0^*)} d_t\right]$$

Note $U_c(c_t, t) = a\delta^t e^{-ac_t}$.

By endowment economy, $C_t^* = d_t^*$

$$P_0 = E_0\left[\sum_{t=1}^{\infty} \frac{a\delta^t e^{-ad_t}}{a e^{-ad_0}} d_t\right]$$

$$P_0 = E_0\left[\sum_{t=1}^{\infty} \delta^t e^{-a(d_t - d_0)} d_t\right]$$

#

3. For the Lucas model with labor income, show that assumptions (6.25) and (6.26) lead to the pricing relationship (6.27) and (6.28).

Derivation of equation (6.27)

From (6.25) $\ln(C^*_{t+1}/C^*_t) = \mu_c + \sigma_c \eta_{t+1}$,

we can write

$$\ln(C^*_{t+2}/C^*_t) = \ln(C^*_{t+2}/C^*_{t+1}) + \ln(C^*_{t+1}/C^*_t)$$
$$= (\mu_c + \sigma_c \eta_{t+2}) + (\mu_c + \sigma_c \eta_{t+1})$$
$$= 2\mu_c + \sum_{i=1}^{2} \sigma_c \eta_{t+i}$$

If we iterate this process, we will have

$$\boxed{\ln(C^*_{t+j}/C^*_t) = j\mu_c + \sum_{i=1}^{j} \sigma_c \eta_{t+j}}$$

* this is applicable for $\ln(d_{t+j}/d_t)$

Next, when we untie $\ln(\cdot)$, we get

$$C^*_{t+j}/C^*_t = \exp(j\mu_c + \sum_{i=1}^{j} \sigma_c \eta_{t+j})$$

From equation (6.24), we know that

$$P_t = E_t\left[\sum_{j=1}^{\infty} \delta^j \left(\frac{C^*_{t+j}}{C^*_t}\right)^{\gamma-1} d_{t+j}\right]$$

$$\frac{P_t}{d_t} = E_t\left[\sum_{j=1}^{\infty} \delta^j \left(\frac{C^*_{t+j}}{C^*_t}\right)^{\gamma-1} \frac{d_{t+j}}{d_t}\right]$$

Plug in c^*_{t+j}/c^*_t and d_{t+j}/d_t into the equation.

$$\frac{P_t}{d_t} = E_t\left[\sum_{j=1}^{\infty}\delta^j \exp\left((\gamma-1)\Big(j\mu_c + \sigma_c\sum_{i=1}^{j}\eta_{t+i}\Big) + j\mu_d + \sigma_d\sum_{i=1}^{j}\varepsilon_{t+i}\right)\right]$$

Next, take the expectation ($E_t[\cdot]$) inside, along with the property of the log-normal distribution

Covariance

$$\frac{P_t}{d_t} = \sum_{j=1}^{\infty}\delta^j \exp\left((\gamma-1)j\mu_c + \frac{1}{2}(\gamma-1)^2\sigma_c^2 j + j\mu_d + \frac{1}{2}\sigma_d^2 j + \frac{1}{2}(2)(\gamma-1)\boxed{\sigma_c\sigma_d\rho} j\right)$$

Note that δ^j can be written as $\exp(j\ln\delta)$, so

$$\frac{P_t}{d_t} = \sum_{j=1}^{\infty}\exp\left(j\ln\delta + (\gamma-1)j\mu_c + \frac{1}{2}(\gamma-1)^2\sigma_c^2 j + j\mu_d + \frac{1}{2}\sigma_d^2 j + \frac{1}{2}(2)(\gamma-1)\sigma_c\sigma_d\rho j\right)$$

$$= \sum_{j=1}^{\infty}\exp\left(j\left[\ln\delta - (1-\gamma)\mu_c + \mu_d + \frac{1}{2}(1-\gamma)^2\sigma_c^2 + \frac{1}{2}\sigma_d^2 - (1-\gamma)\sigma_c\sigma_d\rho\right]\right)$$

Let $\alpha^* = \ln\delta - (1-\gamma)\mu_c + \mu_d + \frac{1}{2}(1-\gamma)^2\sigma_c^2 + \frac{1}{2}\sigma_d^2 - (1-\gamma)\sigma_c\sigma_d\rho = \ln\delta + \alpha$

$$\frac{P_t}{d_t} = \sum_{j=1}^{\infty}\exp(j\alpha^*)$$

By mathematics,

$$\frac{P_t}{d_t} = \sum_{j=1}^{\infty}\exp(j\alpha^*) = \frac{1}{1-e^{\alpha^*}} - 1 = \frac{1}{1-\delta e^{\alpha}} - 1 = \frac{\delta e^{\alpha}}{1-\delta e^{\alpha}}$$

Therefore, $P_t = d_t\dfrac{\delta e^{\alpha}}{1-\delta e^{\alpha}}$ # —— (6.27)

where $\alpha = \mu_d - (1-\gamma)\mu_c + \frac{1}{2}\left[(1-\gamma)^2\sigma_c^2 + \sigma_d^2\right] - (1-\gamma)\rho\sigma_c\sigma_d$ —— (6.28)

4. Consider a special case of the model of rational speculative bubbles discussed in this chapter. Assume that infinitely-lived investors are risk-neutral and that there is an asset paying a constant, one-period risk-free return of $R_f = \delta^{-1} > 1$. There is also an infinitely-lived risky asset with price p_t at date t. The risky asset is assumed to pay a dividend of d_t which is declared at date t and paid at the end of the period, date $t+1$. Consider the price $p_t = f_t + b_t$ where

$$f_t = \sum_{i=0}^{\infty} \frac{E_t\left[d_{t+i}\right]}{{R_f}^{i+1}} \tag{1}$$

and

$$b_{t+1} = \begin{cases} \frac{R_f}{q_t} b_t + e_{t+1} & \text{with probability } q_t \\ z_{t+1} & \text{with probability } 1 - q_t \end{cases} \tag{2}$$

where $E_t\left[e_{t+1}\right] = E_t\left[z_{t+1}\right] = 0$ and where q_t is a random variable as of date $t-1$ but realized at date t and is uniformly distributed between 0 and 1.

4.a Show whether or not $p_t = f_t + b_t$ subject to the specifications in (1) and (2) is a valid solution for the price of the risky asset.

Check whether $\boxed{E_t[b_{t+1}] = \delta^{-1} b_t}$

$$E_t\left[b_{t+1}\right] = \frac{R_f}{\cancel{q_t}} b_t \cancel{q_t} + \cancelto{0}{E_t\left[e_{t+1}\right]} q_t + \cancelto{0}{E\left[z_{t+1}\right]}(1-q_t) = R_f b_t = \delta^{-1} b_t$$

Since, the bubble part satisfies the condition, the solution is valid. #.

4.b Suppose that p_t is the price of a barrel of oil. If $p_t \geq p_{solar}$, then solar energy, which is in perfectly elastic supply, becomes an economically efficient perfect substitute for oil. Can a rational speculative bubble exist for the price of oil? Explain why or why not.

This implies that p_t must be below p_{solar}

From 4.a), we have shown that $E_t[b_{t+1}] = R_f b_t$; so,

$$\lim_{i \to \infty} \delta^{-i} b_t = \begin{cases} +\infty & \text{if } b_t > 0 \\ -\infty & \text{if } b_t < 0 \end{cases}$$

- The price of oil cannot fall to negative infinity. The lowest price possible should be zero.

- Also, the oil price cannot explode to positive infinity since it cannot exceed the price of solar (p_{solar}) which is finite, meaning that the bubble cannot rise above some finite value.

Therefore, the rational bubble does not exist for oil price. #

4.c Suppose p_t is the price of a bond that matures at date $\boxed{T < \infty}$. In this context, the d_t for $t \leq T$ denotes the bond's coupon and principal payments. Can a rational speculative bubble exist for the price of this bond? Explain why or why not.

Using the same reasoning in 4.b),

- The price of bond cannot be negative. Hence, the price surely cannot fall to negative infinity.

- Plus, the price of bond is finite. Since, bond price is d_T at date T, and zero after date T. So, its price surely cannot explode to positive infinity.

Therefore, the rational bubble does not exist for the bond. #

5. Consider an endowment economy with representative agents who maximize the following objective function:

$$\max_{C_s, \{\omega_{is}\}, \forall s, i} E_t \left[\sum_{s=t}^{T} \delta^s u(C_s) \right]$$

where $T < \infty$. Explain why a rational speculative asset price bubble could not exist in such an economy.

With the endowment economy, at date T, $P_T = f_T = d_T$ which implies that $b_T = 0$ in the form $P_T = f_T + b_T$.

Then, the bubble process $E_t[b_{t+1}] = \delta^{-1} b_t$ implies $E_{T-1}[b_T] = E_{T-1}[0] = 0 = b_{T-1}$. With this argrument, we can see that $b_t = 0 \ \forall t < T-1$ as well. #.