EE320 Chapter 5
Nonlinear Model and Differential Calculus in Economic Theory

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1 Review

1.1 Linear function

A general form of a linear function:

$$Y = a + bX$$

- Slope $= \frac{\Delta Y}{\Delta X} = \frac{Y_2 - Y_1}{X_2 - X_1} =$ b

  Pick any two points on the function and plug it into the formula: (x_1, y_1) and $(x_2, y_2) \Rightarrow = \frac{y_2 - y_1}{x_2 - x_1} = \frac{a + bx_2 - a - bx_1}{x_2 - x_1} = \frac{b(x_2 - x_1)}{x_2 - x_1} = b$

- Interpretation: If X increases by 1 unit, Y will increases by 2 units
- Slope of a linear function is constant

1.2 Non-Linear function

One of the example of an non-linear function is:

$$Y = aX^2 + bX + C$$

- Slope is <u>not</u> constant, as slope changes at each point
- To find slope of non-linear function, pick two points on the function: $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$, then

  Slope $= \frac{f(x_0+h) - f(x_0)}{x_0 + h - x_0} = \frac{f(x_0+h) - f(x_0)}{h}$

- Slope of f(x) at x_0: keep h as small as possible and get the tangent line

  $f^{'}(x) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \frac{dy}{dx} = \frac{df(x)}{dx}$

  Example: $f(x) = ax^2 + bx + c$

  $\frac{df(x)}{dx} = 2ax + b \quad \Rightarrow$ slope for any points of f(x)

2 Rule of differentiation

The derivative can be derived using the following rule of differentiation:

1. $f(x) = c \Rightarrow$
$$\frac{df(x)}{dx} = 0$$
2. $f(x) = Ax^n \Rightarrow$
$$\frac{df(x)}{dx} = A \cdot n \cdot x^{n-1}$$
3. $f(x) = Ae^x \Rightarrow$
$$\frac{df(x)}{dx} = Ae^x$$
4. $f(x) = \ln x \Rightarrow$
$$\frac{df(x)}{dx} = \frac{1}{x}$$
5. $f(x) = a^x \Rightarrow$
$$\frac{df(x)}{dx} = a^x \ln a$$
6. $f(x) = \log_a x \Rightarrow$
$$\frac{df(x)}{dx} = \frac{1}{x \ln a}$$

For two or more function of the same variable

7. $h(x) = f(x) \pm g(x) \Rightarrow$
$$h'(x) = \frac{dh(x)}{dx} = f'(x) \pm g'(x)$$

8. Product Rule: $y = f(x)g(x) \Rightarrow$

$$\frac{dy}{dx} = f(x)g'(x) + g(x)f'(x)$$

Example: $y \;=$

$\frac{dy}{dx} \;=$

$=$

9. Quotient Rule: $y = \frac{f(x)}{g(x)} \Rightarrow$

$$\frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Example: $y \;=\; \frac{x^2+2x+3}{x^3+7}$

$\frac{dy}{dx} \;=$

$=$

10. Chain Rule: $y = f(u)$ and $u = g(x) \Rightarrow$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\begin{aligned}
\text{Example: } y &= ln(x^2+1) \\
u &= x^2+1 \\
\frac{dy}{dx} &=
\end{aligned}$$

$$\begin{aligned}
\text{Example: } y &= ln(x^2+1)^5 \\
u &= (x^2+1)^5 \\
\frac{dy}{dx} &=
\end{aligned}$$

3 Non Differential functions

Differentiability of a function: a function is differentiable if it is continuous and smooth. "Continuity" is a necessary condition, while "smoothness" is a sufficient condition. Let us study each concept.

3.1 Continuity

A function y =f(x) is <u>continuous</u> at x_0 if the following are true:

1) $f(x_0)$ is defined.

2) $\lim_{x\to x_0} f(x)$ exists $\Leftrightarrow \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x)$

3) $\lim_{x\to x_0} f(x) = f(x_0)$

Example: f(x) is discontinuous at $x_0 = 5$ if

$$\lim_{x \to x_0^-} f(x) = 1 \quad \neq \quad \lim_{x \to x_0^+} f(x) = 20$$

Example: $$\begin{array}{rcll} g(x) & = & x^2 & \text{if } \; x < 2 \\ & = & x+1 & \text{if } \; x \geq 2 \end{array}$$

$\lim_{x \to x_0^-} g(x) =$

$\lim_{x \to x_0^+} g(x) =$

$g(x)$ is .... at $x_0 = 2$

3.2 Smoothness

A function is smooth when it is differentiable everywhere. It is differentiable at point x_0 if

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}, \text{ where } h \neq x_0, \text{ exists}$$

Example: $f(x) = |x - 2| + 1$

To find whether this function is differentiable or not, we need to find whether the derivation at some x exists, so we study x where $x \to 2^-$ or where x approaches 2 (which can be probably at $1.\dot{9}$)

$\lim_{x \to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x \to 2^-} \frac{|x-2|+1-|2-2|-1}{x-2} = \lim_{x \to 2^-} \frac{|x-2|}{x-2} = \frac{-(x-2)}{x-2} = -1$

Then study at $x \to 2^+$

$\lim_{x \to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x \to 2^+} \frac{|x-2|+1-|2-2|-1}{x-2} = \lim_{x \to 2^+} \frac{|x-2|}{x-2} = \frac{(x-2)}{x-2} = 1$

As we can see, the limits of the slope at 2 are not equal, which means it does not exist and thus this function is not smooth.

4 Example in Economics

4.1 Derivative and Marginality

In economics, marginality indicates rate of change - how much the value of the function changes as the choice variable (independent variable) increases by one unit.

$$\begin{aligned}
\text{Example: TP} &\Rightarrow \text{AP} &= \tfrac{\text{TP}}{\text{L}} &\Rightarrow \tfrac{d\text{TP}}{d\text{L}} &= \text{MP} \\
\text{TR} &\Rightarrow \text{AR} &= \tfrac{\text{TR}}{\text{Q}} &\Rightarrow \tfrac{d\text{TR}}{d\text{Q}} &= \text{MR} \\
\text{TC} &\Rightarrow \text{AC} &= \tfrac{\text{TC}}{\text{Q}} &\Rightarrow \tfrac{d\text{TC}}{d\text{Q}} &= \text{MC} \\
\text{C} &\Rightarrow \text{APC} &= \tfrac{\text{C}}{\text{Y}} &\Rightarrow \tfrac{d\text{C}}{d\text{Y}} &= \text{MPC} \\
\text{S} &\Rightarrow \text{APS} &= \tfrac{\text{S}}{\text{Y}} &\Rightarrow \tfrac{d\text{S}}{d\text{Y}} &= \text{MPS}
\end{aligned}$$

4.2 Relation among the total, the average and the marginal functions

4.2.1 Revenue

Total Revenue = P $\times$ Q

Demand function: Q^d = a -bP $\Rightarrow$ P = $\frac{a}{b} - \frac{1}{b}Q^d$

Consider a monopoly which is the only firm selling this goods and has market power to control the amount of goods supplied to the whole market. The firm will set the quantity supplied and price by considering consumers' demand

$\text{TR} = P(Q) \cdot Q \Rightarrow \text{AR} = \frac{P(Q) \cdot Q}{Q} = P(Q)$

$$\mathrm{MR} \;=\; \frac{d\mathrm{TR}}{dQ} = \frac{d\Big(P(Q)\times Q\Big)}{dQ}$$

$$\mathrm{MR} \;=\; \mathrm{AR}\left(1+\frac{1}{\varepsilon_p^d}\right)$$

$$\begin{array}{lll} \text{If } \mid \varepsilon_p^d \mid < 1 & \Rightarrow & MR < 0 \;\; \text{inelastic} \\ \quad\; \mid \varepsilon_p^d \mid > 1 & \Rightarrow & MR < 0 \;\; \text{elastic} \\ \quad\; \mid \varepsilon_p^d \mid = 1 & \Rightarrow & MR < 0 \;\; \text{unitary elastic} \end{array}$$

4.2.2 Product

For short run production,
$$Q = f(L)$$
$$\mathrm{AP} = \frac{f(L)}{L} \Rightarrow$$
$$\frac{d\mathrm{AP}}{dL} =$$

$$\mathrm{MP} = \frac{df(L)}{dL} = f'(L)$$

If (I) $\mathrm{MP} > \mathrm{AP} \Rightarrow \frac{dAP}{dL} > 0$

(II) $\mathrm{MP} = \mathrm{AP} \Rightarrow \frac{dAP}{dL} = 0$

(III) $\mathrm{MP} < \mathrm{AP} \Rightarrow \frac{dAP}{dL} < 0$

4.2.3 Cost

Total Cost = TFC + TVC = a + f(Q) = C(Q)

$$\text{AC} = \frac{C(Q)}{Q} \Rightarrow$$

$$\frac{d\text{AC}}{dQ} =$$

$$\text{MC} = \frac{d}{dQ}C(Q)$$

If (I) MC < AC $\rightarrow \frac{dAC}{dQ} < 0$

II) MC = AC $\rightarrow \frac{dAC}{dQ} = 0$

(III) MC > AC $\rightarrow \frac{dAC}{dQ} > 0$

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