EE320 Lecture Note
Chapter 5: Nonlinear Model and Differential Calculus in Economic Theory

- Recall <u>linear function</u> : Y = a + bX

  $\frac{\Delta Y}{\Delta X}$ = b

  ex. $\frac{\Delta Y}{\Delta X}$ = 2 if X $\uparrow$ by 1 unit, Y $\uparrow$ by 2 units.

  Slope for linear function is constant.

- For <u>non-linear function</u>: Y = $aX^2 + bX + C$

Slope is <u>not</u> constant, so slope will change.

To find slope of non-linear function,

slope = $\frac{f(x_0+h)-f(x_0)}{x_0+h-x_0} = \frac{f(x_0+h)-f(x_0)}{h}$

slope of f(x) at x_0, keep h as small as possible, get the tangent line

$f^{'}(x) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \frac{dy}{dx} = \frac{df(x)}{dx}$

<u>ex.</u> $f(x) = ax^2 + bx + c$

$\frac{df(x)}{dx}$ = 2ax +b   slope for any points of f(x)

1 Rule of differentiation

A function of one variable

	f(x)	f'(x)
1	c	0
2	Ax^n	Anx^{n-1}
3	Ae^x	Ae^x
4	lnx	$\frac{1}{x}$
5	a^x	$a^x lna$
6	$log_a x$	$\frac{1}{xlna}$

Two or more function of the same variable

7. $h(x) = f(x) \pm g(x) h'(x) = f'(x) \pm g'(x)$

8. Product Rule

y = f(x)g(x) $\quad \frac{dy}{dx} = f(x)g'(x) + g(x)f'(x)$

$$\begin{aligned} \text{ex. } y &= (x^2+1)(x^4+4x+1) \\ \frac{dy}{dx} &= (x^2+1)\frac{d}{dx}(x^4+4x+1) + (x^4+4x+1)\frac{d}{dx}(x^2+1) \\ &= (x^2+1)(4x^3+4)(x^4+4x+1)(2x) \end{aligned}$$

9.Quotient Rule

$y = \frac{f(x)}{g(x)} \quad \frac{dy}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

$$\begin{aligned} \text{ex. } y &= \frac{x^2+2x+3}{x^3+7} \\ \frac{dy}{dx} &= \frac{(x^3+7)\frac{d}{dx}(x^2+2x+3) - (x^2+2x+3)\frac{d}{dx}(x^3+7)}{(x^3+7)^2} \\ &= \frac{(x^3+7)(2x+3) - (x^2+2x+3)(3x^2)}{(x^3+7)^2} \end{aligned}$$

10. Chain Rule

$$\begin{aligned} \text{ex. } y &= ln(x^2+1) \\ u &= x^2+1 \\ \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{du}{dx} = \frac{1}{x^2+1} \cdot 2x \end{aligned}$$

ex. $y = ln(x^2+1)^5$

$u = (x^2+1)^5$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{(x^2+1)^5} 5(x^2+1)^4(2x) = \frac{10x}{(x^2+1)}$

2 Non Differential functions

Differentiability of a function: a function is differentiable if it is continuous and smooth.

"Continuity" is a necessary condition.

"Smoothness" is a sufficient condition.

2.1 Continuity

A function y =f(x) is <u>continuous</u> atx_0 if the following are true:

1) $f(x_0)$ is defined.

2) $\lim_{x\to x_0} f(x)$ exists $\leftrightarrow \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x)$

3) $\lim_{x\to x_0} f(x) = f(x_0)$

ex. f(x) is discontinuous at $x_0 = 5$

$\lim_{x\to x_0^-} f(x) = 1 \quad \neq \quad \lim_{x\to x_0^+} f(x) = 20$

$$\begin{aligned} ex.g(x) &= x^2 && \text{if} \quad x < 2 \\ &= x+1 && \text{if} \quad x \geq 2 \end{aligned}$$

$\lim_{x\to x_0^-} g(x) = 4 \neq \lim_{x\to x_0^+} g(x) = 3 g(x)$ is discontinuous at $x_0 = 2$

2.2 Smoothness

A function is smooth when it is differentiable everywhere.

A function is differentiable at point $x_0 iff'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}; h \neq x_0$

ex. f(x) = | x-2 | +1

$\lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-} \frac{|x-2|+1-|2-2|-1}{x-2} = \lim_{x\to 2^-} \frac{|x-2|}{x-2} = \frac{-(x-2)}{x-2} = -1$

$\lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+} \frac{|x-2|+1-|2-2|-1}{x-2} = \lim_{x\to 2^+} \frac{|x-2|}{x-2} = \frac{(x-2)}{x-2} = 1$

3 Example in Economics

3.1 Derivative and Marginality

In economics, marginality indicates rate of change - how much the value of the function changes as the choice variable (independent variable) increases by one unit.

$$
\begin{aligned}
\text{ex. } TP &\Rightarrow AP &= \tfrac{TP}{L} &\Rightarrow \tfrac{dTP}{dL} &= MP \\
TR &\Rightarrow AR &= \tfrac{TR}{Q} &\Rightarrow \tfrac{dTR}{dQ} &= MR \\
TC &\Rightarrow AC &= \tfrac{TC}{Q} &\Rightarrow \tfrac{dTC}{dQ} &= MC \\
C &\Rightarrow APC &= \tfrac{C}{Y} &\Rightarrow \tfrac{dC}{dY} &= MPC \\
S &\Rightarrow APS &= \tfrac{S}{Y} &\Rightarrow \tfrac{dS}{dY} &= MPS
\end{aligned}
$$

3.2 Relation among the total, the average and the marginal functions

✓ Total Revenue = P × Q

Demand function : Q^d = a -bP → P = $\frac{a}{b} - \frac{1}{b}Q^d$

As a monopoly, firm needs to set the price by considering consumers'demand

TR = P(Q)· Q , AR = [P(Q)-Q]/Q = P(Q)

$$
\begin{aligned}
MR = \tfrac{dTR}{dQ} = QP'(Q) + P(Q)\tfrac{dQ}{dQ} = P(Q) + Q\tfrac{dP(Q)}{dQ} &= AR + Q \cdot \tfrac{dP(Q)}{dQ} \times \tfrac{P(Q)}{P(Q)} \\
&= AR + \tfrac{Q}{P(Q)} \cdot \tfrac{dP(Q)}{dQ} \cdot P(Q) \\
&= AR + \tfrac{P(Q)}{\varepsilon_p^d} \\
MR &= AR\left(1 + \tfrac{1}{\varepsilon_p^d}\right)
\end{aligned}
$$

$$
\begin{aligned}
\text{If } |\, \varepsilon_p^d \,| < 1 &\Rightarrow MR < 0 \text{ inelastic} \\
|\, \varepsilon_p^d \,| > 1 &\Rightarrow MR < 0 \text{ elastic} \\
|\, \varepsilon_p^d \,| = 1 &\Rightarrow MR < 0 \text{ unitary elastic}
\end{aligned}
$$

✓ Total Product

For SR production,

$$Q = f(L)$$

$$AP = \frac{f(L)}{L} \Rightarrow \frac{dAP}{dL} = \frac{Lf'(L)-f(L)\frac{dL}{dL}}{L^2} = \frac{MP}{L} - \frac{AP}{L} = \frac{MP-AP}{L}$$

$$MP = \frac{df(L)}{dL} = f'(L)$$

$$I)\ MP > AP \Rightarrow \frac{dAP}{dL} > 0$$

$$II)\ MP = AP \Rightarrow \frac{dAP}{dL} = 0$$

$$III)\ MP < AP \Rightarrow \frac{dAP}{dL} < 0$$

✓ Total Cost = TFC + TVC = a + f(Q) = C(Q)

AC = $\frac{C(Q)}{Q}$ $\frac{dAC}{dQ} = \frac{QC'(Q)-C(Q)\frac{dQ}{dQ}}{Q^2} = \frac{MC}{Q} - \frac{AC}{Q} = \frac{MC-AC}{Q}$

$MC = \frac{d}{dQ}C(Q)$

$$I)MC < AC \to \frac{dAC}{dQ} < 0$$

$$II)MC = AC \to \frac{dAC}{dQ} = 0$$

$$III)MC > AC \to \frac{dAC}{dQ} > 0$$