# Constrained optimization

$$\left\{\boxed{\begin{array}{lll} \max \ (\min) & f(x,y) \\ \text{s.t.} & g(x,y) = c \end{array}}\right.$$

$\max (\min)\ f(x,y) \rightarrow$ objective.

s.t. $g(x,y) = c \rightarrow$ single constraint

- Subsitution method
- LaGrange method.

obictvin $f^n$ + $\lambda\,[c - g(x,y)]$

$\rightarrow \mathcal{L} = f(x,y) + \lambda\,[c - g(x,y)]$

FOC. $L_x = L_y = L_\lambda = 0$

$$\left.\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0\right\} \Rightarrow (x^*, y^*, \lambda^*)$$

↳ $sol^n$ for the constrained optimization Problem

~~Tangenty con~~

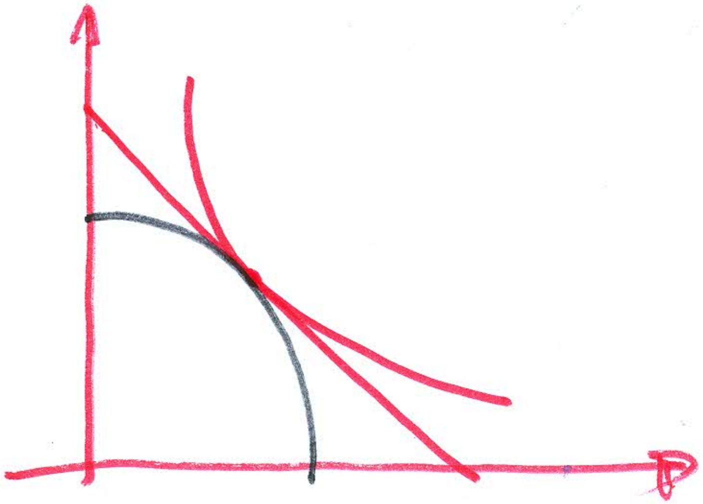
$$\rightarrow \frac{\partial L}{\partial x} = \frac{\partial f}{\partial x} + \lambda\left(-\frac{\partial g}{\partial x}\right) = 0$$

$$\rightarrow \frac{\partial L}{\partial y} = \frac{\partial f}{\partial y} + \lambda\left(-\frac{\partial g}{\partial y}\right) = 0$$

$$\rightarrow \frac{\partial L}{\partial \lambda} = c - g(x,y) = 0$$

$(x^*, y^*, \lambda^*)$

$$\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}}$$



slope of the level set of your objective f$^{\underline{n}}$ = slope of your constraint set

Tangent Condition

max $x \cdot y$ s.t. $x + y = c$ ⟵ sol$^{n}$ to

**CP** constraint optimization Problem.

$$\mathcal{L} = x \cdot y + \lambda\left[c - (x+y)\right]$$ ⟵ sol$^{n}$ to this problem.

find stationary points of "L"

**F.O.C.**

$$\frac{\partial L}{\partial x} \overset{L_x}{=} y + \lambda(-1) = 0 \quad \text{— ①} \qquad \Rightarrow y = \lambda$$

$$\frac{\partial L}{\partial y} \overset{L_y}{=} x + \lambda(-1) = 0 \quad \text{— ②} \qquad x = \lambda$$

$$\frac{\partial L}{\partial \lambda} \overset{L_\lambda}{=} c - (x+y) = 0 \quad \text{— ③}$$

$x$
$y$
$\lambda$

3 equations

$$c - (\lambda + \lambda) = 0 \longrightarrow \lambda = \frac{c}{2}$$

$$x = c/2 \; ; \; y = \frac{c}{2}$$

$(x^*, y^*, \lambda^*)$ is stationary point of L

$\left(\frac{c}{2}, \frac{c}{2}, \frac{c}{2}\right)$

$\max \quad x \cdot y$

$$y = \lambda$$

s.t. $\quad \underline{x + y} = c$

$$\mathcal{L} = x \cdot y + \lambda [c - (x+y)]$$

$\underline{\text{F.O.C.}}$

$$\left.\begin{aligned} \mathcal{L}_x &= y + \lambda(-1) = 0 \\ \mathcal{L}_y &= x + \lambda(-1) = 0 \\ \mathcal{L}_\lambda &= c - (x+y) = 0 \end{aligned}\right\} \quad y = x$$

$$x + x = c \rightarrow x = \frac{c}{2} \; ; \; y = \frac{c}{2}$$

$$? \longleftarrow \lambda = c/2$$

## Optimal Value $f^n$

look objective $f^n$ $F(x, y)$

$x(c)$ ; $y(c)$ → constrained optimizers

$f^*(c) = f(x(c), y(c))$ → optimal Value $f^n$

→ optimized Value of your objective $f^n$

"c"

$$\frac{df^*}{dc} = \lambda \Rightarrow$$

$\lambda$ measures the sensitivity of your optimal value $f^n$ with respect to a unit relaxation in the constraint c-value.

$$f^*(c) = f(x(c), y(c))$$

$$\frac{df^*}{dc} = \frac{d f(x(c), y(c))}{dc}$$

$$= \underbrace{\frac{\partial f}{\partial x}}_{\lambda \cdot \frac{\partial g}{\partial x}} \cdot \frac{dx}{dc} + \underbrace{\frac{\partial f}{\partial y}}_{\lambda \cdot \frac{\partial g}{\partial y}} \cdot \frac{dy}{dc}$$

$$= \lambda \cdot g_x \cdot \frac{dx}{dc} + \lambda \cdot g_y \cdot \frac{dy}{dc} = \lambda \underbrace{\left[ g_x \frac{dx}{dc} + g_y \frac{dy}{dc} \right]}_{=1}$$

$$\mathcal{L}_x = \mathcal{L}_y = 0$$

$$\mathcal{L}_x \quad \frac{\partial f}{\partial x} - \lambda \cdot \frac{\partial g}{\partial x} = 0$$

$$\mathcal{L}_y \quad \frac{\partial f}{\partial y} - \lambda \cdot \frac{\partial g}{\partial y} = 0$$

~~g(x(c),~~ $g(x(c), y(c)) = c$

$$\frac{dg(x(c), y(c))}{dc} = \frac{dc}{dc} = 1$$

$$\Downarrow$$

$$\frac{\partial g}{\partial x} \cdot \frac{dx}{dc} + \frac{\partial g}{\partial y} \cdot \frac{dy}{dc} = 1$$

$$\frac{df^*}{dc} = \lambda$$

# Check for the Second Order Condition.

2nd order derivative matrix of the Lagrange f²

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda [c - g(x, y)]$$

$$\Rightarrow \begin{bmatrix} \mathcal{L}_{\lambda\lambda} & \mathcal{L}_{\lambda x} & \mathcal{L}_{\lambda y} \\ \mathcal{L}_{x\lambda} & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ \mathcal{L}_{y\lambda} & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{bmatrix}$$

$\mathcal{L}_\lambda = c - g(x, y)$

$\mathcal{L}_x = f_x - \lambda g_x$

$\mathcal{L}_y = f_y - \lambda \cdot g_y$

Bordered Hessian matrix

$$\bar{H} = \begin{bmatrix} 0 & -g_x & -g_y \\ -g_x & f_{xx} - \lambda g_{xx} & f_{xy} - \lambda g_{xy} \\ -g_y & f_{xy} - \lambda \cdot g_{xy} & f_{yy} - \lambda \cdot g_{yy} \end{bmatrix}$$

**Theorem:** $(x^*, y^*, \lambda^*)$ satisfied F.O.C.

max if $|\bar{H}| > 0$

min if $|\bar{H}| < 0$

$$\bar{H} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$$|\bar{H}| = 0 + 1 + 1 + 0 + 0 + 0$$

$$= 2 > 0$$

$$\left.\begin{aligned} x &= \frac{c}{2} \\ y &= \frac{c}{2} \end{aligned}\right\} \text{truly max point}$$

obj. $f(x,y) = \underline{2y + x}$

$x=0,\ y=-1,\ \lambda=-1$

$x=0,\ y=1,\ \lambda=1$

Constraint $y^2 + xy = 1$

Set up the

Step 1 Lagrange f$^n$

$$\mathcal{L} = 2y + x + \lambda\left[1 - y^2 - xy\right]$$

$\frac{1 \text{ point}}{1{,}000}$

Step 2 Finding the stationary point sol$^n$ for the Lagrange f$^n$

$\Rightarrow y = 1/\lambda$

$L_x = 1 - \lambda y = 0 \quad -①$

$L_y = 2 - 2\lambda y - x\lambda = 0 \quad -②$

$L_\lambda = 1 - y^2 - xy = 0 \quad -③$

Step 3 Solve for $(x^*, y^*, \lambda^*)$ satisfies the three F.O.C.

Hessian → $\ominus$, $\oplus$

Profit f^n

$$TR - TC \rightarrow \underline{+}4(Q_A + Q_B) + F$$

$$TR = TR_A + TR_B$$

$$TR_A = P_A Q_A \qquad TR_B = P_B \cdot Q_B$$

3rd degree price discrimination

constraint-free optimized level of profit

$$\Pi = (100 - 2Q_A)Q_A + (50 - 0.5Q_B)Q_B - 4(Q_A + Q_B) - F$$

$$\frac{\partial \Pi}{\partial Q_A} = \frac{\partial \Pi}{\partial Q_B} = 0$$

$$\frac{\partial \Pi}{\partial Q_A} = 100 - 4Q_A - 4 = 0$$

$$96 = 4Q_A$$

$$Q_A = 24 \rightarrow P_A = 52$$

$$\frac{\partial \Pi}{\partial Q_B} = 50 - Q_B - 4 = 0$$

$$Q_B = 46 \rightarrow P_B = 27$$

optimized level of profit

$$\Pi^* = (24)(52) + (46)(27) - 4(24 + 46) - F = 2210 - F$$

Profit f^n but feasible

$$Q_A + \overset{=g}{Q_B} = \overset{=c}{60} \text{ units}$$

Set the Lagrange $f^z$

$\Pi + \lambda[c - g(x,y)]$ Profit is objective

$$\mathcal{L} = (100 - 2Q_A) \cdot Q_A + (50 - 0.5Q_B)Q_B - 4(Q_A + Q_B) - F + \lambda\left[60 - (Q_A + Q_B)\right]$$

$\lambda = 96 - 4Q_A$

$$\mathcal{L}_A = 100 - 4Q_A - \cancel{2}4 - \lambda = 0$$

$\lambda = 46 - Q_B$

$$\mathcal{L}_B = 50 - Q_B - 4 - \lambda = 0$$

$Q_A + Q_B = 60$ *

$$\mathcal{L}_\lambda = 60 - Q_A - Q_B = 0$$

$Q_A$, $Q_B$, $\lambda$

$\lambda = 8$

$$96 - 4Q_A = 46 - Q_B = \lambda$$

$\lambda =$

$$4Q_A - Q_B = 50 \; *$$

$Q_A = \dfrac{110}{5} = 22$

$Q_B = 38$

$$\bar{H} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -4 & 0 \\ -1 & 0 & -1 \end{bmatrix} \quad |\bar{H}| = 0 + 0 + 0 + 4 + 0 + 1 = 5 > 0$$

Bordered Hessian

Confirm

Constrained Sol$^n$

$P_A$ $56 = 100 - 2(22) \longleftarrow Q_A = 22$

$P_B$ $31 = 50 - 0.5(\overset{19}{38}) \longleftarrow Q_B = 38$

Constrained maximizer

$\pi$ constraint less than

Constrain free Sol$^n$

$Q_A = 24$ $Q_B = 46$ $\pi^* = 2210 - F$

$P_A = 52$ $P_B = 27$

$\lambda = f^*(c)$

Object $f^{\underline{n}}$ $\rightarrow$ optimizm (c)

optimized level of profit

optimized value $f^{c} = f^*(c)$

$c \Rightarrow$ Quota / 1 unit

60 $\rightarrow$ 61

$$\frac{d f^*(c)}{d \underline{c}} = \lambda$$

relax 1 unit value for c

$\Rightarrow f^*(c)$ change by $\lambda$

$\lambda$ approach works when $\underline{\Delta C \text{ is small}}$

initial situation is Total Quota is $100$

$$\frac{\partial \Pi^*}{\partial C} = \lambda \quad 0$$

$\to Q_A$

$Q_B$

$\downarrow$

Constrain the output to

$C \uparrow \to \Pi^* \downarrow$

$\lambda \quad -$

Total Quta = $\boxed{60}$

$\pi^* = 2210 - F$

Quota = ~~61~~ 70

$\pi^*$ Constrained $= \cancel{2?.56 + (31)38}$

$= 56.22 + 31 \cdot 38$

$- 4(60) - F$

$\lambda = 68$

how $\pi^*$

$= \cancel{2???}F$

$2178 - F$

$= \underline{2170 - F}$ ✓

Sub optimality

δu 2250-F

utility $f^n$

$\rightarrow$ $U(x, y)$ $\rightarrow$ Ordinary Demand

Marshallian Demand.

$$u(x_1, x_2, x_3, \dots x_n)$$
$$\rightarrow P_1x_1 + P_2x_2 + \dots P_nx_n = M$$

n goods

Budget = $M$

Price of $x$ = $P_x$

Price of $y$ = $P_y$

$$\max_{(x, y)} \; u(x, y)$$

2 goods

LaGrange $f^n$

subject to — s.t. Budget Set

$$P_x \cdot x + P_y \cdot y = M$$

$$\mathcal{L} = u(x,y) + \lambda[M - P_x \cdot X - P_y Y]$$

$x, y$ $\qquad \mathcal{L}_x = \mathcal{L}_y = \mathcal{L}_\lambda = 0$

$x, y, \lambda$ Satisfy 3 Equations at the same time

$$\mathcal{L}_x = \underbrace{\frac{\partial u}{\partial x}}_{MU_x} + \lambda(-P_x) = 0 \quad — ①$$

$$\mathcal{L}_y = \underbrace{\frac{\partial u}{\partial y}}_{MU_y} + \lambda(-P_y) = 0 \quad — ②$$

$$\mathcal{L}_\lambda = M - P_x X - P_y Y = 0 \quad — ③$$

Slope of BC

IC is tangent to the budget set

$$MU_x = \lambda \cdot P_x$$

$$MU_y = \lambda \cdot P_y$$

slope of the IC

$$\frac{MU_x}{MU_y} = \frac{P_x}{P_y}$$

MRS of x for y

Indirect Utility f$^{n}$

is the optimized level of

utility f$^{n}$.

$$U(X,y) \longrightarrow \begin{matrix} X^*(M, P_x, P_y) \\ y^*(M, P_x, P_y) \end{matrix}$$

Plugging $x^*, y^*$ into $U \Rightarrow \underline{U(x^*, y^*)}$

Indirect Utility f$^{n}$.

$$P_x \cdot x + \cancel{P_y}\left(\frac{b}{a} \cdot \frac{P_x}{\cancel{P_y}} \cdot x\right) = M$$

$$a \cdot P_x \cdot x + b \cdot P_x \cdot x = a \cdot M$$

$$(a+b) P_x \cdot x = a \cdot M$$

$$P_x \cdot x = \frac{a}{a+b} \cdot M$$

$$\frac{\partial x}{\partial M} = \frac{a}{a+b} \cdot \frac{1}{P_x} > 0$$

$$x = \frac{a}{a+b} \cdot \frac{M}{P_x}$$

$$\frac{\partial x}{\partial P_x} = \frac{a}{a+b} \cdot \left(-\frac{M}{P_x^2}\right) < 0$$

$y = ?$

$$y = \frac{b}{a} \cdot \frac{a}{a+b} \cdot \frac{M}{P_x} \cdot \frac{P_x}{P_y} = \frac{b}{a+b} \cdot \frac{M}{P_y}$$

# Consumer theory (311)

1. • Derive Individual Demand (static) → max U

Alfred Marshall → Marshallian Demand (Ordinary)

John Hicks → Hicksian Demand (Compensated demand) → min expenditure

2. • Consumer-Leisure Problem

→ Consume → work

3. Intertemporal Consumption Problem

$$u(x,y) = x^{a} y^{b}$$

$$MU_x = a x^{a-1} y^{b}$$

$$MU_y = b x^{a} y^{b-1}$$

Substitution method

$$\frac{①}{②} = \frac{a}{b} \cdot \frac{1}{x} \cdot y = \frac{P_x}{P_y}$$

$a+b$ : degree of HM.

$$① \quad a x^{a-1} \cdot y^{b} = \lambda \cdot P_x \rightarrow \mathcal{L}_x = 0$$

$$② \quad b x^{a} \cdot y^{b-1} = \lambda \cdot P_y \rightarrow \mathcal{L}_y = 0$$

$$P_x \cdot x + P_y y = M \rightarrow \mathcal{L}_\lambda = 0$$

$$x = \frac{a}{a+b} \cdot \frac{M}{P_x} \qquad y = \frac{b}{a} \cdot \frac{P_x}{P_y} \cdot x$$

$$\frac{a}{b} \cdot \frac{y}{x} = \frac{P_x}{P_y}$$

$$P_x x + P_y y = M$$

$\lambda$ in UMP is

the marginal ultity of income

$$\frac{\partial v}{\partial M} = \lambda$$

M $\Delta M = 1$

$v = \lambda$

$\Uparrow$ U strictly increasing

always $\lambda > 0$

## Marshallian (ordinary) demand

$\mathcal{L} = U(x, y) + \lambda[M - P_x x - P_y y]$

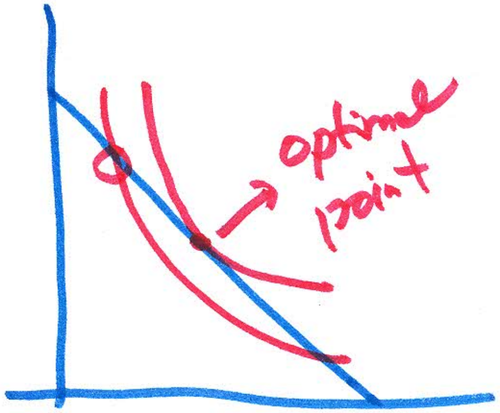
$$\max_{x,y} \; U(x, y)$$ → utility

$$s.t. \qquad P_x x + P_y Y = M$$ → B.S.

Using the LaGrange method, the three FOCs $(L_x = L_y = L_\lambda = 0)$ yield us the following two equations.

(i) From $L_\lambda = 0$, we would know that: $P_x x + P_y Y = M$

(ii) From $L_x = L_y = 0$, we would obtain that $\frac{MU_x}{MU_y} = \frac{P_x}{P_y}$




MRS    Price ratio

Example $U(x,y) = x^a y^b$ → Cobb-Douglas form, HM degree $a+b$

Demand $f^n$ ←

$$\begin{cases} x^* = x^*(P_x, P_y, M) = \dfrac{a}{a+b}\dfrac{M}{P_x} \\ \\ y^* = y^*(P_x, P_y, M) = \dfrac{b}{a+b}\dfrac{M}{P_y} \end{cases}$$

$\frac{\partial x^*}{\partial P_x} = \frac{a}{a+b}\left(-\frac{M}{P_x^2}\right) < 0$

$\frac{\partial y^*}{\partial P_x} = 0$

$\frac{\partial x^*}{\partial P_y}$ ; $\frac{\partial y^*}{\partial P_x} > 0$ ; substitute products

Property of $x^*$ and $y^*$

(p1) Downward sloping in its own price (law of demand: true if U is strictly increasing)

(p2) No cross price effect. (Be careful!. only this specific example of utility function)

(p3) Increasing in income, i.e. normal goods (again!, true under this specific example)

(p4) Homogeneity of degree ZERO in $P_x, P_y, M$. (Always true.)

a. ✓ Price ratio remains the same, so (ii) holds under new prices } $x^*, y^*$ stay the same!

b. ✓ Everything double up, budget set doesn't change, i.e. no money illusion.

$t \cdot P_x$, $tM$ <br> $t P_y$ } ⇒ $(t \cdot P_x) \cdot x + (t \cdot P_y) \cdot y = t \cdot M$

## The indirect utility function

- Optimal value function: $f^*(c)$, i.e. plugging the value of constrained optimizer into the objective function.

$\frac{\partial f^*}{\partial c} = \lambda$

- Indirect utility function is the maximized (optimized) level of utility (under the utility maximization problem)

f : utility

c : Income

$$v = U(x^*, y^*) = v(P_x, P_y, M)$$

$\frac{\partial v}{\partial M} = \lambda$

**The Lamda as the marginal utility of income:**

Having relaxed the constraint value imposed by the level of income (M), our maximized level of utility would be increasing by $\lambda^*$.

$$\frac{\partial v(P_x, P_y, M)}{\partial M} = \lambda^* = \lambda^*(P_x, P_y, M) = \frac{MU_x(x^*, y^*)}{P_x}$$

By the assumption that U is strictly increasing, $\lambda^* > 0$.

This is to capture the idea of "the more, the better".

Our example: $u(x,y) = x^a y^b \rightarrow \left(\frac{a}{a+b}\cdot\frac{M}{P_x}\right)^a \left(\frac{b}{a+b}\cdot\frac{M}{P_y}\right)^b$

$$v = \left[\frac{a}{a+b}\frac{M}{P_x}\right]^a \left[\frac{b}{a+b}\frac{M}{P_y}\right]^b = \underbrace{\left[\frac{a}{a+b}\frac{1}{P_x}\right]^a \left[\frac{b}{a+b}\frac{1}{P_x}\right]^b}_{} M^{a+b} = K \cdot M^{a+b}$$

$$\frac{\partial v(P_x, P_y, M)}{\partial M} = K(a+b)M^{a+b-1} = \lambda^*$$

One thing that you have to do is to check the second order condition.

- I will skip this. But, it's by default that you need to check the SOC and confirm the solution that you obtain from the FOCs.

$|\bar{H}| \rightarrow$ max $> 0$

$|\bar{H}| \rightarrow$ min $< 0$

## Hicksian demand (compensated demand)

Taking the level of (target) utility as given, we search for the bundle of consumption that minimizes the expenditure.

$\mathcal{L} = P_x X + P_y \cdot y + \lambda[\bar{u} - u(x,y)]$

Mathematically speaking, the problem is to solve:

$$\min \quad P_x x + P_y Y$$

s.t. $U(x,y) = \bar{u}$ where $\bar{u}$ is a given level of utility attained.

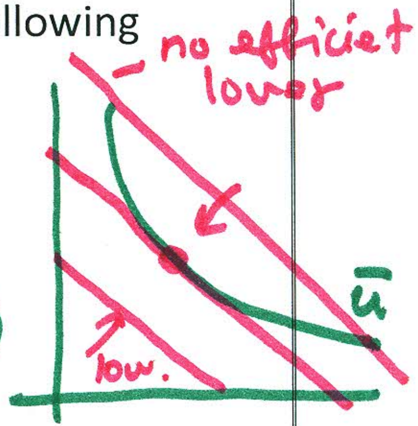
① $\mathcal{L}_x \Rightarrow P_x - \lambda \cdot MU_x = 0$ , ② $\mathcal{L}_y = P_y - \lambda \cdot MU_y = 0$ ; ③ $L_\lambda = \bar{u} - u(x,y) = 0$

Using the LaGrange method, the three FOCs $(L_x = L_y = L_\lambda = 0)$ yield us the following two equations.

(i) From $L_\lambda = 0$, we would know that: $\boxed{U(x,y) = \bar{u}}$

(ii) From $L_x = L_y = 0$, we would obtain that $\dfrac{MU_x}{MU_y} = \dfrac{P_x}{P_y}$

① + ②



Example: $U(x,y) = x^a y^b$

$$\frac{MU_x}{MU_y} = \frac{a}{b} * \frac{y}{x} = \frac{P_x}{P_y} \rightarrow y = \frac{b}{a}\frac{P_x}{P_y}x$$

Target utility $\bar{u}$

$$x^a y^b = \bar{u} \rightarrow x^a \left(\frac{b}{a}\frac{P_x}{P_y}x\right)^b = \bar{u}$$

$$\rightarrow x^{a+b}\left(\frac{b}{a}\cdot\frac{P_x}{P_y}\right)^b = \bar{u}$$

$$x^{a+b} = \bar{u}\left(\frac{a}{b}\frac{P_y}{P_x}\right)^b \rightarrow x^* = x^*(\bar{u}, P_x, P_y) = \bar{u}^{\frac{1}{a+b}}\left(\frac{a}{b}\frac{P_y}{P_x}\right)^{\frac{b}{a+b}}$$

$$\bar{u}^{\frac{1}{a+b}} \cdot \left(\frac{a}{b}\cdot\frac{P_y}{P_x}\right)^{\frac{b}{a+b}}$$

$$y^* = y^*(\bar{u}, P_x, P_y) = \bar{u}\left(\frac{b}{a}\frac{P_x}{P_y}\right)^{\frac{a}{a+b}}$$

$$MU_x = \frac{\partial u}{\partial x} = a x^{a-1} \cdot y^b$$

$$MU_y = \frac{\partial u}{\partial y} = b x^a y^{b-1} \Rightarrow y$$

$$\frac{MU_x}{MU_y} = \frac{P_x}{P_y} = \frac{a x^{a-1} \cdot y^b}{b x^a \cdot y^{b-1}} \quad -①$$

$$\frac{y}{x}$$

$$x^a y^b = \bar{u} \quad -②$$

$$① \quad \frac{y}{x} \cdot \frac{a}{b} = \frac{P_x}{P_y}$$

$$\rightarrow y = \frac{b}{a} \cdot \frac{P_x}{P_y} \cdot X$$

**Property:**

(p1): Compensated demand curve is decreasing in own price.

(p2): Homogeneity of degree zero in "prices"

**The expenditure function**

- The analogue version of the indirect utility function in the UMP problem.

minimized

- Expenditure function is the minimized level of expenditure for attaining the targeted level of utility.

$$e(\underline{P_y}, \underline{P_x}, \underline{\bar{u}}) = P_x x^* + P_y y^* = P_x \bar{u} \underline{\left(\frac{a}{b}\frac{P_y}{P_x}\right)}^{\frac{b}{a+b}} + P_y \bar{u} \underline{\left(\frac{b}{a}\frac{P_x}{P_y}\right)}^{\frac{a}{a+b}}$$

$x^*, y^*$ in the ~~expenidi~~ expenditure equation.

$$e = P_x \cdot \left[\bar{u} \cdot \left(\frac{a}{b} \cdot \frac{P_y}{P_x}\right)^b\right]^{\frac{1}{a+b}}$$

$$+ P_y \cdot \left[\bar{u} \left(\frac{b}{a} \cdot \frac{P_x}{P_y}\right)^a\right]^{\frac{1}{a+b}}$$

**CONSUMPTION-LEISURE problem**

Household consumes, but consumption depends on two sources of income.

e.g. Earning income v.s. endowed income.

Earning income is derived from work.

People love recreation, and hates working. But, after all, they have to work.

Consumption-leisure model explains the decision problem that represents the trade-off between consumption and leisure (opposite of work)

(i) consuming more and making more money from work.

(ii) consuming less and do nothing for your work.

L ; C

**Mathematical formulation:**

objective $f^n$

$$\max_{C,L} \overset{++}{U(C,L)}$$

$C\uparrow \rightarrow u\uparrow$ ; $L\uparrow \rightarrow u\uparrow$

C is consumption (in units of gods)

L is leisure (in hours)

Total spending on your consumption goods = Total income.

s.t. $PC = w(24 - L) + x$ (Suppose a daily problem)

w is the wage per hour.

x is the endowed income.

working = $24-L$

$PC = 24W - WL + X$

earned income $= w \cdot$ Working Hours $= w(24-L)$

Budget set: $PC + wL = 24w + x$

Budget set in the problem.

LaGrange function $\mathcal{L} = U(C,L) + \lambda[\underbrace{24w + x}_{\text{Constrained value}} - \underbrace{PC - wL}_{g(.)}]$

FOC. $\mathcal{L}_c = \mathcal{L}_L = \mathcal{L}_\lambda = 0$

From $\mathcal{L}_\lambda = 0$,

(i) $PC + wL = 24w + x$

From $\mathcal{L}_c = \mathcal{L}_L = 0$, we obtain that

$$(ii)\ U_C(C, L) = \lambda P$$

$$(iii)\ U_L(C, L) = \lambda w$$

Using (ii) and (iii), we yield that :

$$\frac{U_L(C, L)}{U_c(C, L)} = \frac{w}{P}$$

**Rearranging terms:**

$$U_L(C,L) = \frac{w}{P} U_c(C,L)$$

**Trade-off between Leisure and Consumption**

What is the cost an hour reduction in the leisure?

What is the benefit from a reduction in the leisure by an hour?

- Earning more income by $w$
- Each additional income can be used to purchase the consumption goods by $\frac{w}{P}$.
- Each unit of additional intake consumption goods yields us $U_c(C,L)$
- $\frac{w}{P} U_c(C,L)$ is the marginal benefit that arises from one more hour of working.

$$\max \quad u = 8\sqrt{c} + l$$

$$\text{s.t.} \quad P_c \cdot C + wl = y_0 + 24 \cdot w$$

$$\frac{4}{\sqrt{c}} = \frac{P_c}{w}$$

$$\sqrt{c} = \frac{4 \cdot w}{P_c} \Rightarrow C^* = \frac{16w^2}{P_c^2}$$

<u>LaGrange $f^n$.</u>

$$\mathcal{L} = 8\sqrt{c} + l + \lambda[y_0 + 24 \cdot w - P_c \cdot C - wl]$$

$$\frac{4}{\sqrt{c}} - \left(\frac{1}{w}\right) \cdot P_c = 0$$

$$\begin{pmatrix} C^*, L^* \\ \lambda^* \end{pmatrix}$$

$$\frac{\partial \mathcal{L}}{\partial c^*} = \frac{4}{\sqrt{c}} - \lambda \cdot P_c = 0 \qquad \text{— (i)}$$

$$\lambda = \frac{1}{w}$$

$$\frac{\partial \mathcal{L}}{\partial l} = 1 - \lambda \cdot w = 0 \qquad \text{— (ii)}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = y_0 + 24 \cdot w - P_c \cdot C - wl = 0 \qquad \text{— (iii)}$$

$$y_0 + 24 \cdot w - P_c \cdot \left(\frac{16w^2}{P_c^2}\right) - w \cdot l = 0$$

## Example (Fall 2015, Final)

Joe's utility function depends on consumption and leisure, and it is given by:

$$U(c,l) = 8\sqrt{c} + l,$$

where $c$ is consumption $l$ is leisure. Each day Joe works $n$ hours and spends the rest of the time on leisure; that is, $l + n = 24$. Suppose further that Joe's total income is the sum of a fixed non-wage income ($Y_0$) and a wage income ($Y_n$). The wage income ($Y_n$) is equal to $wn$, where $w$ is the hourly wage rate. If Joe spends all of his income on the consumption, his budget constraint becomes:

$$p_c c = Y_0 + wn = Y_0 + w(24 - l),$$

which can also be written as:

$$p_c c + wl = Y_0 + 24w.$$

**3.1.** ***(10 points)*** Use Lagrange method to derive the optimal values of $c^*$ and $l^*$ in terms of the parameters $p_c$, $w$, and $Y_0$. State the condition under which the consumer demands some leisure (i.e., $l^* > 0$).

**3.2.** ***(5 points)*** Based on your answer in (3.1), derive the impact of an increase in the hourly wage ($w$) on the demand for leisure ($l^*$). Interpret its economic meaning.

**3.3.** ***(10 points)*** Suppose now that the hourly wage rate ($w$) is \$24, the price of consumption ($p_c$) is \$12, and the fixed non-wage income ($Y_0$) is 384. Use Lagrange method to determine the levels of $c^*$ and $l^*$ that maximizes Joe's utility subject to the budget constraint, and determine the level of constrained maximum utility.

**3.4.** ***(5 points)*** Verify your answers in (3.3) by using the second-order sufficient condition.

Suppose that the financial market exists, with $(y_t, y_{t+1})$ as the level of income in the two periods, and $r$ is the interest rate.

today tomorrow

saving income how much you choose for the consumption

$$B_t = y_t - c_t$$ (Can $B_t$ have a negative value?)

$$c_{t+1} = y_{t+1} + B_t(1+r)$$

Consolidating the two constraints, we yield that

$$(1+r)c_t + c_{t+1} = (1+r)y_t + y_{t+1}$$

Rearranging term: $$\underbrace{c_t + \frac{c_{t+1}}{1+r}}_{g} = \underbrace{y_t + \frac{y_{t+1}}{1+r}}_{\text{Constraint Value}}$$

**Resource constraints in terms of the present value adjustment.**

consumption for tomorrow

## A MODEL OF INTERTEMPORAL CONSUMPTION

Consumer theory. static Problem

$$\max_{x,y} \; u(x, y) \quad \text{s.t.} \quad P_x x + P_y y = M$$

consumption for today

The problem of allocating your resource for consumption in two periods

- Today v.s. Future
- Yong v.s. Old → no working income

↳ working, working ability

People have a preference relationship defined over the bundle/level of consumption in two periods, $U(c_t, c_{t+1})$ → u is concave

current current

future consumption

↳ smoothing consumption motive

People might have different level of income over time.

- High when young, low when hold.. or vice versa.

How the resource can be allocated (over time) depends on institutional structure of market economy, i.e. whether the financial market exists?

- Financial market. Allowing for savings and borrowings. → vehicle that helps smooth consumption over time

Example: $U = \ln(c_t) + \beta\ln(c_{t+1})$ $\longrightarrow$ Time additive preference

$\beta = time\ discount\ factor.$ (today is better than tomorrow.) $0 < \beta < 1$

periodic ultility $f^n$ = $h(.)$

$$\boxed{\frac{MU_t}{\beta MU_{t+1}} = 1+r}$$ 311

$c_t^*, c_{t+1}^*$

Set up LaGrang $f^n$

$MU_t = \beta(1+r) \cdot MU_{t+1}$

$$\mathcal{L} = \ln c_t + \beta \ln c_{t+1} + \lambda \left[ y_t + \frac{y_{t+1}}{1+r} - c_t - \frac{c_{t+1}}{1+r} \right]$$

$[c_t]: \frac{1}{c_t} - \lambda = 0 \Rightarrow$ MU of current consumption $= \lambda$

$[c_{t+1}]: \frac{\beta}{c_{t+1}} - \frac{\lambda}{1+r} = 0 \Rightarrow \beta \cdot$ mu of future consumption $= \frac{\lambda}{1+r}$

$[\lambda]:$ Intertemporal Budget constraint

What happen to the current consumption and saving/borrowing if interest rate increases?

$$\frac{1}{C_t} = \lambda \longrightarrow \underline{C_t = \frac{1}{\lambda}} = \frac{1}{1+\beta}\left(y_t + \frac{y_{t+1}}{1+r}\right)$$

$$\frac{\beta}{C_{t+1}} = \frac{\lambda}{1+r} \longrightarrow \underline{C_{t+1} = \frac{\beta(1+r)}{\lambda}}$$

$$C_{t+1} = \left(\frac{1}{\lambda}\right)\beta(1+r) = \frac{\beta(1+r)}{1+\beta}\left(y_t + \frac{y_{t+1}}{1+r}\right)$$

$$C_t + \frac{C_{t+1}}{1+r} = y_t + \frac{y_{t+1}}{1+r}$$

$$\frac{1}{\lambda} + \frac{\frac{\beta\cancel{(1+r)}}{\lambda}}{\cancel{1+r}} = \overbrace{y_t + \frac{y_{t+1}}{1+r}}^{\text{Total of P.V. of your income}}$$

$$\frac{1}{\lambda}\left[1+\beta\right] = y_t + \frac{y_{t+1}}{1+r} \Rightarrow \frac{1}{\lambda} = \frac{1}{1+\beta}\cdot\left(\begin{array}{c}\text{Total}\\ \text{PV of}\\ \text{income}\end{array}\right)$$

| Consumer | Producer |
|---|---|
| UMP max $u(x,y)$ s.t. $P_x X + P_y Y = M$ | OMP max $f(K,L)$ s.t. $w \cdot L + rK =$ Cost budget |
| Demand x, y (ordinary demand) Marshallion D | → you would get the demand for K and L. |
| eMP min $P_x X + P_y Y$ $u(x,y) = \bar{u}$ | CMP min $wL + rK$ s.t. $f(K,L) = \bar{Q}$ |
| Demond x, y : Compensated demand Curve. Hicksian D. | → you get demand for K and L. Read it yourself! |

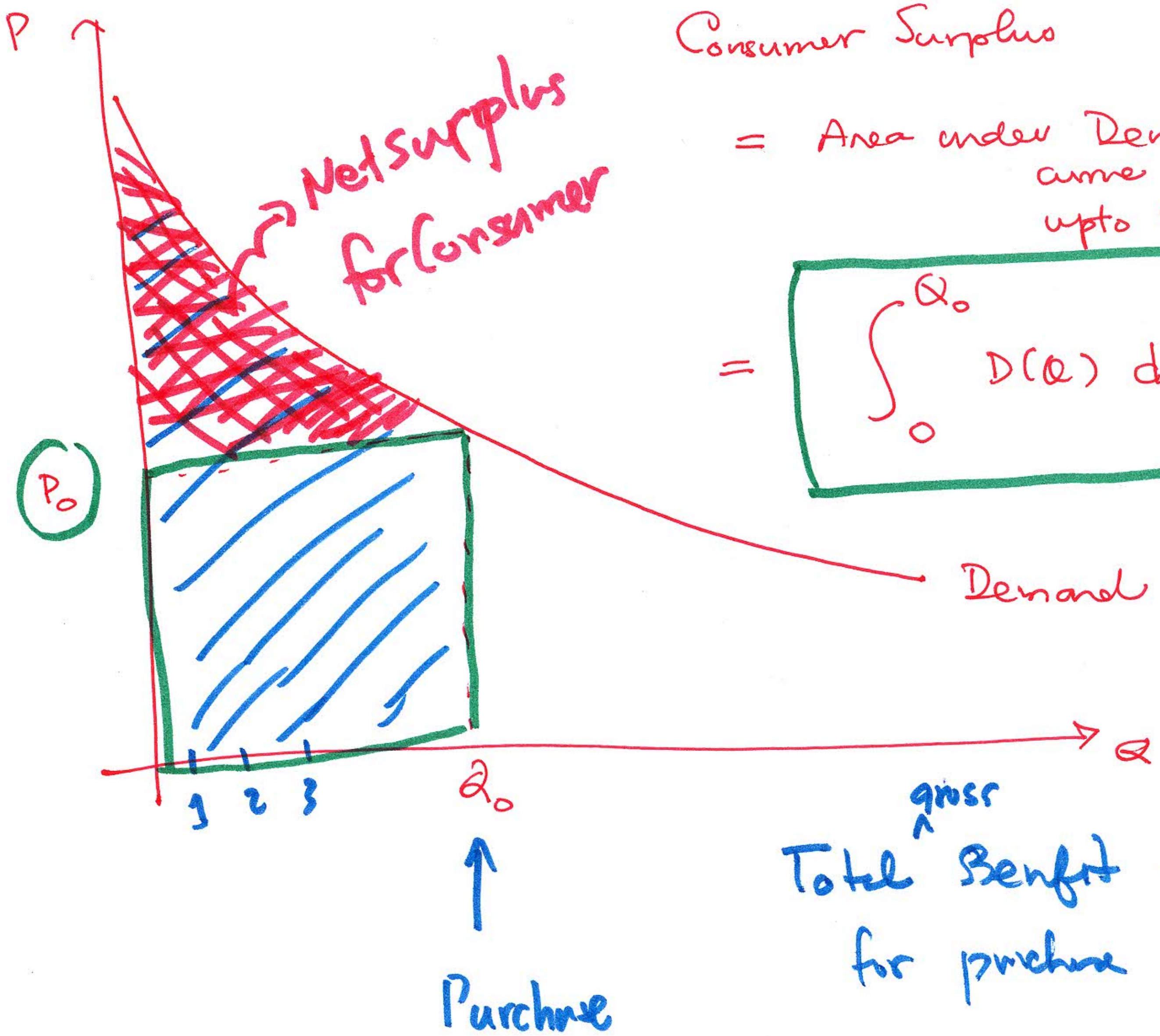
# Integration

Marginal function $\Longrightarrow$ Total function

Welfare Analysis.

- Area under the curve.

$MU = P$ ; $u$ is concave / $u$ is strictly law of diminishing MU



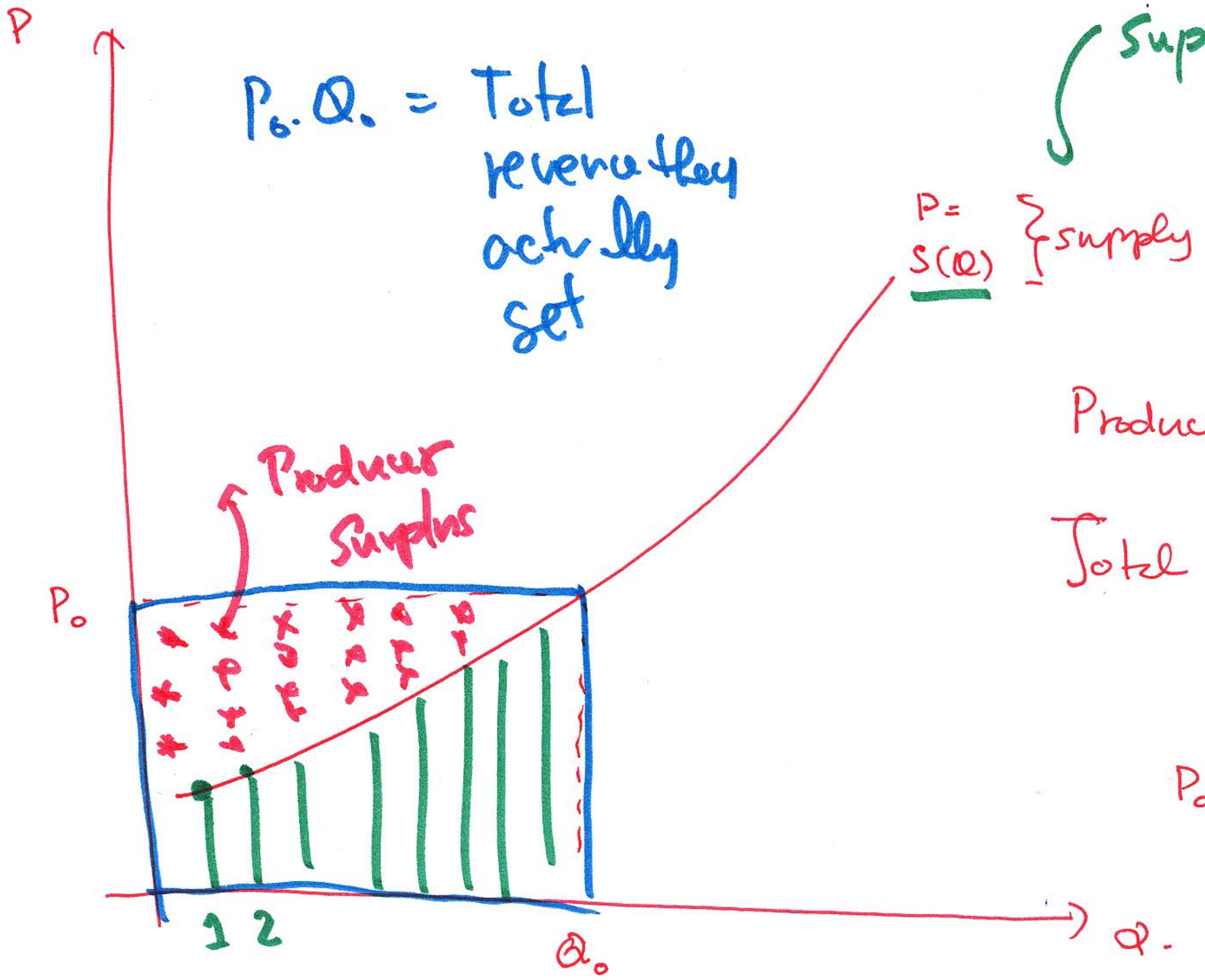

Consumer Surplus

= Area under Demand curve upto $Q_0$ − Total payment up to $Q_0$

$$= \boxed{\int_0^{Q_0} D(Q)\, dQ - P_0 \cdot Q_0} \quad \underline{CS}$$

Demand ⇒ $\underline{P = D(Q)}$

Total ^gross Benefit for Consumer for purchase upto $Q_0$ units




Supply $\underline{\underline{Q}}$: $P = \underline{mc}$

mc is increasing

Producer Surplus =

Total receipts for the sell up to $Q_0$ −

Area under the supply.

$$P_0 Q_0 - \int_0^{Q_0} S(Q)\,dQ$$

Total minimum require level of revenue for producer.

## Welfare for Market Equilibrium.

Total Welfare (CS+PS) is highest; under PC

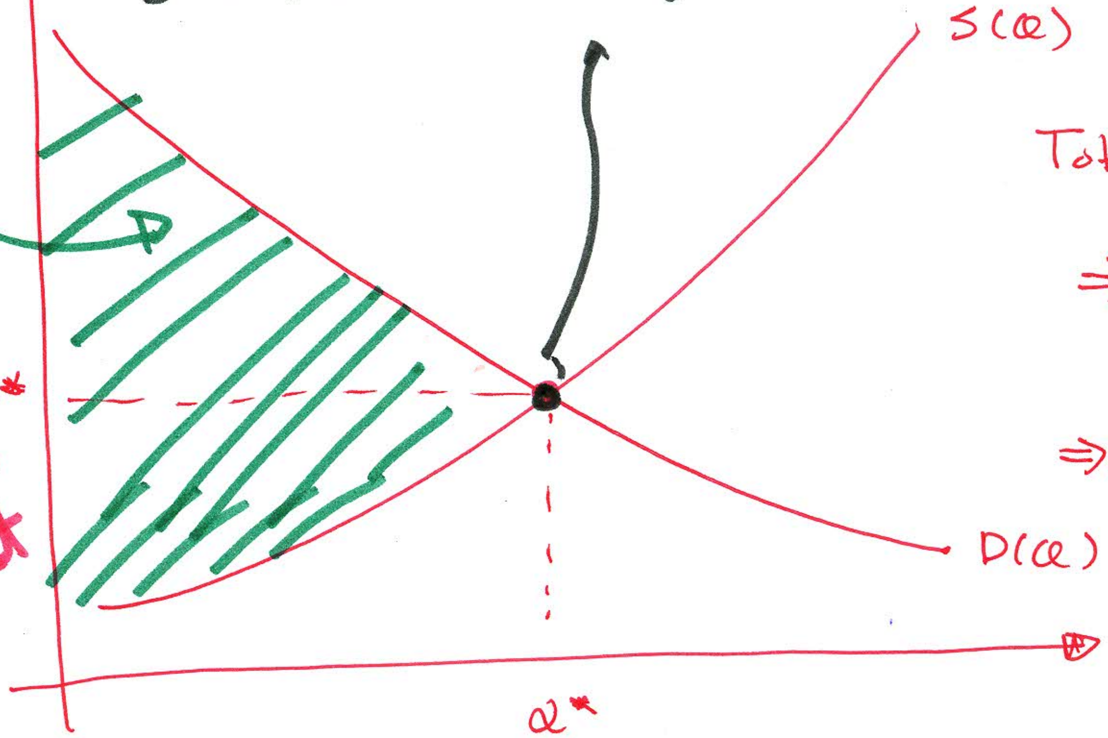
Total Welfare = CS + PS

Equ^n output D = S (Perfectly Competitive market)

Welfare Triangle

Area of welfare Triangle is biggest

↳ Benchmark first-best




Total Surplus at $Q^*$ (Equ^m)

$\Rightarrow \underset{\text{at } Q^*}{CS} + \underset{\text{at } Q^*}{PS}.$

$\Rightarrow CS: \int_0^{Q^*} D(Q)\,dQ - P^* Q^*$

$PS: P^* Q^* - \int_0^{Q^*} S(Q)\,dQ$

Step $(P^*, Q^*)$

$\therefore \; TS = \int_0^{Q^*} D(Q)\,dQ - \int_0^{Q^*} S(Q)\,dQ$

$= \int_0^{Q^*} [\, D(Q) - S(Q) \,]\, dQ.$

TS: Area below D above S bounded by Q=0 and $Q^*$

4

property

Under Perfectly competitive market.

- Welfare Triangle is the largest.
- Efficient in the sense of the two surpluses combined. (ignore about distributional aspect).

Suboptimality of the welfare implication.

(i) Market Distortion (Tax, subsidy)

(ii) Monopoly distortion. ( $Q_{monopoly} < Q_{perfect\ comp}$ ).

(iii) Frictions / Externalities

$Max^{unco}$ $-$ $+$ $-$

$Max^{Con}$ $+$