

Practice problem set 4

Single-variable optimization

EE320 Semester 2/2015

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Question 1 Total product function

Suppose that $TP(L) = 120L^2 - L^3$, where TP is the total output. Answer the following questions

- a. Find the expression for AP and MP.

$$AP(L) = 120L - L^2 \text{ and } MP(L) = 240L - 3L^2$$

- b. Find the value of L that maximizes AP, MP and TP, respectively.

$$\text{Maximize AP} \Rightarrow FOC : AP'(L) = 120 - 2L = 0 \implies L = 60$$

$$SOC : AP''(60) = -2 < 0$$

$$\text{Maximize MP} \Rightarrow FOC : MP'(L) = 240 - 6L = 0 \implies L = 40$$

$$SOC : (MP''(40) = -6 < 0)$$

Maximize TP \Rightarrow

$$FOC : TP'(L) = MP(L) = 240L - 3L^2 = 0 \implies L = 0, 80.$$

$$SOC : (TP''(L) = 240 - 6L \implies TP''(80) < 0.$$

- c. Define the domain set of L that justifies the production function.

One should be noted that (i) L must be greater than or equal to zero. (negative L doesn't make sense. Second, firm will not produce in the region that L is greater than 80 units. When L exceeds 80 units, TP decreases.

Question 2

Let the total cost function be:

$$TC(Q) = 2Q^2 - 8Q + 10.$$

a) Determine whether $TC(Q)$ is a convex or concave function.

Ans. $TC'(Q) = 4Q - 8. \Rightarrow TC''(Q) = 4 > 0$. Thus, $TC(Q)$ is a convex function.

b) Find the quantity Q^* that minimizes the total cost.

Ans. $TC'(Q) = 4Q - 8 = 0. \Rightarrow Q^* = 2$.

c) Verify that $TC(Q^*)$ is the lowest cost by using the second derivative test.

Ans. $TC''(2) = 4 > 0$. Thus, $TC(2) = 2$ is the minimum.

Question 3

Let the demand function be given by $Q = 60 - 2P$ where P is the unit price and Q is the amount of quantity. Assume that the production function of a firm is given by $Q = 5\sqrt{L}$ where L is the number of workers hired. Consider the following questions.

a. Suppose that wage is \$50 per each worker hired. This is fixed, regardless to how many workers hired by a firm. Write down the equation that summarizes the total cost of hiring labor by “L” workers.

$$Cost = W * L = 50L$$

b. Normally, our cost function is written in terms of Q . Based on “a”, can you find a way to represent the total cost equation in the terms of Q , rather than L ?

$$\text{Note that } Q = 5\sqrt{L} \Rightarrow Q^2 = 25L \Rightarrow L = \frac{1}{25}Q^2 .$$

$$Cost = 50L = 50 \left(\frac{1}{25}Q^2 \right) = 2Q^2$$

- c. Use the cost function in “b”, and solve for the level of profit-maximizing output/price. Verify your answer

First, rewrite demand into p-equal form. That is, $P = \frac{60-Q}{2}$

$$\pi(Q) = \frac{60-Q}{2}Q - 2Q^2$$

$$\text{FOC: } \pi'(Q) = 30 - Q - 4Q = 0$$

$$Q = 6.$$

SOC: $\pi''(Q) = -5 < 0 \Rightarrow$ This warrants that $Q = 6$ is actually the profit-maximizing level of output.

- d. At the level of profit-maximizing output, how many workers should the firm hire?

$$Q = 6 \implies L = \frac{1}{25}(6)^2 = 1.44 \text{ units.}$$

Question 4 Deriving the market supply

Suppose the cost function of a representative firm can be given by:

$C(Q) = 3Q^2 + 5Q + 75$ where Q is the level of output produced. Answer the following question

- a. Derive the expression for MC, AVC, and ATC.

$$MC(Q) = 6Q + 5$$

$$AVC(Q) = \frac{TKC}{Q} = 3Q + 5$$

$$ATC(Q) = 3Q + 5 + \frac{75}{Q}$$

- b. Find the level of Q that results in the lowest total cost.

That is, when $MC = 0$.

$$MC(Q) = 6Q + 5 = 0 \implies Q = -5/6.$$

However, it is not possible to have negative Q.

As you can see from the equation, MC is always positive. That means, $C(Q)$ is a strictly increasing function for positive Q.

As a result, the lowest cost would be when $Q = 0$. That's firm only pays for the fixed cost.

- c. Find the supply equation of the representative firm. Make sure you specify the range of price that justifies your equation as the one representing the supply equation.

$$P = MC(Q)$$

$$P = 6Q + 5$$

When $Q = 0$, $P = 5$. That is, firm needs to earn at least \$5 if it would stay active in the market.

- d. If there are 60 identical firms in the market, derive the market supply curve.

$$Q = \frac{P-5}{6} \implies Q^s_M = 60 * \left(\frac{P-5}{6}\right) = 10(P-5) \text{ when } P \text{ is greater than or equal to } 5. \quad P = 5 + \frac{Q}{10}$$

Question 5

Given the following function

$$f(x) = 2x^3 + 8x^2 - 32x - 50$$

- a. Find the critical value(s) of x and the corresponding stationary value(s) of $f(x)$.

$$\text{Ans. } f'(x) = 6x^2 + 16x - 32 = 0 \implies x^* = -4, 4/3$$

- b. Evaluate whether the stationary value(s) found in part a) are relative maxima or minima or inflection points by using *the first-derivative test*.

$$\text{Ans. } f(-4) \text{ is a maximum because } f'(-5)=38 > 0 \text{ and } f'(-3)=-26 < 0.$$

$$f(4/3) \text{ is a minimum because } f'(1)=-10 < 0 \text{ and } f'(2)=24 > 0.$$

Question 6

A competitive firm receives a price p for each unit of its output, and pays a price w for each unit of its only variable input. It also incurs set up costs of

F. Its output from using x units of variable input is $f(x) = \sqrt{x}$. Determine the firm's revenue, cost, and profit functions.

Ans. For $x > 0$, $R = p\sqrt{x}$; $C = wx + F$; $\pi(x) = p\sqrt{x} - wx - F$.
 $\pi'(x) = 0$ when $w = p/2\sqrt{x} \Rightarrow x = p^2/4w^2$. Check: $\pi''(x) < 0$

Question 7

The price a firm obtains for a commodity varies with demand Q according to the formula $P(Q) = 18 - 0.006Q$. Total cost is $C(Q) = 0.004Q^2 + 4Q + 4500$.

a. Find the firm's profit and the value Q which maximizes profit.

Ans. $\pi(Q) = Q.P(Q) - C(Q) = -0.01Q^2 + 14Q - 4500 \Rightarrow Q^* = 700$

b. Find a formula for the elasticity of $P(Q)$ w.r.t. Q , and find the particular value Q^* of Q at which the elasticity is equal to -1.

Ans. This is the price elasticity w.r.t. demand ($Q=Q_d$):

$$E_d = \frac{Q}{P} \frac{dP}{dQ} = \frac{Q}{18-0.006Q} (-0.006) = -1 \Rightarrow Q^* = 1500$$

c. Show that the marginal revenue is 0 at Q^* .

Ans. $MR = 18 - 0.012Q \Rightarrow$ at $Q^* = 1500$, $MR = 0$.

Question 8 Monopoly and Subsidy program

A monopolist firm faces the market demand equation given by $P = 150 - 0.5Q$ and operates under a technology with cost function given by $TC = 100 + 3Q + 7Q^2$. Consider the following questions

a. By using the derivative method, find the level of profit-maximizing output and price. Verify that your answer is the correction solution that results in maximized profit.

$$MR = 150 - Q$$

$$MC = 3 + 14Q$$

$$\text{FOC: } MR = MC$$

$$150 - Q = 3 + 14Q$$

$$Q = \frac{147}{15}$$

SOC:

$$\pi'' = MR' - MC' < 0$$

$$MR' = -1$$

$$MC' = 14$$

$$\pi'' = -15 < 0$$

So, $Q = \frac{147}{15}$ is the level of profit-maximizing output.

Continue with all the information given above, but now add another assumption to the questions. That is, we now assume that government subsidizes the monopolist for \$3 for each unit of output.

- b. Write the cost function of the monopolist when subsidization is taken into account.

$$TC = 100 + 3Q + 7Q^2 - 3Q = 100 + 7Q^2.$$

- c. Find the level of profit-maximizing output and price under the subsidy program.

Redo the same as in (a), but now we use the new TC as obtained in (b).

FOC:

$$MR = MC$$

$$150 - Q = 14Q$$

$$Q = 10.$$

SOC. $\pi'' = -15 < 0$. Thus, $Q = 10$ is the level of profit-maximizing output.

Given this Q, price would be \$145. (This can be obtained by plugging Q into the demand equation.)

Question 9: (midterm 1/2015)

A monopolist is operating under a cost function given by,

$$TC = 2Q^3 - 18Q^2 + 93Q + 50,$$

where Q is the quantity of output.

Suppose that the market demand for the goods produced by this monopolist is given by: $P = 81 - Q^2$, where P is the price per unit of output.

Consider the following problems.

- a. **(5 points)** Determine the *absolute* value of the price elasticity of demand when $P = \$54$.
- b. **(5 points)** Without solving for the optimization, determine the revenue-maximizing level of output.
- c. **(15 points)** Derive the profit function, and solve for the profit-maximizing level of output. Verify your answer by using the second-order derivative.
- d. **(5 points)** Would the monopolist change the level of output if government taxes the monopolist based on the profit level that he earns?

No solution given