Chapter 3: Linear Space (Vector Spaces): Part 2

1 Linear Independence

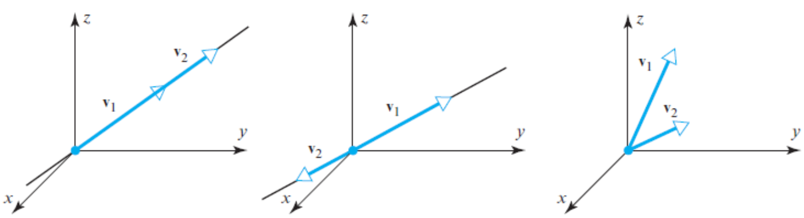
In this section we will consider the question of whether the vectors in a given set are interrelated in the sense that one or more of them can be expressed as a linear combination of the others. This is important to know in applications because the existence of such relationships often signals that some kind of complication is likely to occur.

Definition 1.1. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a set of two or more vectors in a vector space V , then S is said to be a **linearly independent** set if no vector in S can be expressed as a linear combination of the others.

A set that is not linearly independent is said to be **linearly dependent**.

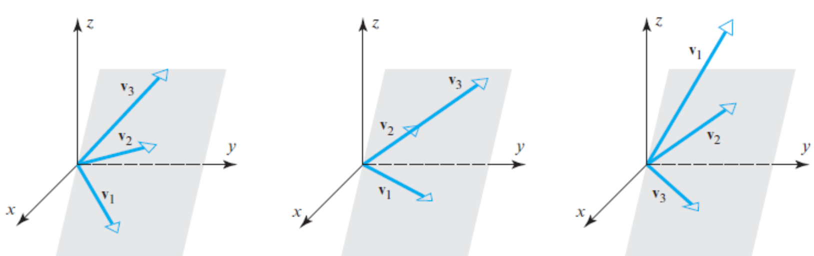
1.1 A Geometric Interpretation of Linear Independence in $\mathbb{R}^2$ and $\mathbb{R}^3$

- Two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ are linearly independent if and only if they do not lie on the same line when they have their initial points at the origin. Otherwise one would be a scalar multiple of the other.



(a) Linearly dependent (b) Linearly dependent (c) Linearly independent

- Three vectors in $\mathbb{R}^3$ are linearly independent if and only if they do not lie in the same plane when they have their initial points at the origin. Otherwise at least one would be a linear combination of the other two.



(a) Linearly dependent (b) Linearly dependent (c) Linearly independent

There are many ways to determine whether a set is linearly independent. An efficient way to do this is to use the following theorem.

Theorem 1.1. A nonempty set $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r\}$ in a vector space V is linearly independent if and only if the only coefficients satisfying the vector equation

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_r\mathbf{v}_r = 0$$

are $k_1 = 0, k_2 = 0, \ldots, k_r = 0$.

Proof:

Example 1.1. The most basic linearly independent set in $\mathbb{R}$ is the set of standard unit vectors:

Example 1.2. Determine whether the vectors

$$\mathbf{v}_1 = (1, 2, 2, -1), v_2 = (4, 9, 9, -4), v_3 = (5, 8, 9, -5)$$

in $\mathbb{R}^4$ are linearly dependent or linearly independent.

Example 1.3. Determine whether the vectors

$$\mathbf{v}_1 = (1, -2, 3), v_2 = (5, 6, -1), v_3 = (3, 2, 1)$$

are linearly independent or linearly dependent in $\mathbb{R}^3$.

Example 1.4. (Exercise) Prove the following statements.

(a) A finite set that contains $\mathbf{0}$ is linearly dependent.

(b) A set with exactly one vector is linearly independent if and only if that vector is not $\mathbf{0}$.

(c) A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.

Theorem 1.2. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_r\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$, then S is linearly **dependent**.

Proof:

1.2 Linear Independence of Functions

Example 1.5. Linear Independence of Two Functions

- The functions $f_1 = x$ and $f_2 = \sin(x)$ are linearly independent vectors in $F(-\infty, \infty)$ since neither function is a scalar multiple of the other.
- The two functions $g_1 = \sin(2x)$ and $g_2 = \sin(x)\cos(x)$ are linearly dependent because the trigonometric identity $\sin(2x) = 2\sin(x)\cos(x)$ shows that g_1 and g_2 are scalar multiples of each other

Example 1.6. Show that the polynomials

$$1, x, x^2, ..., x^n$$

form a linearly independent set in P_n.

Example 1.7. Determine whether the polynomials

$$p_1 = 1 - x, \quad p_2 = 5 + 3x - 2x^2, \quad p_3 = 1 + 3x - x^2$$

are linearly dependent or linearly independent in P_2.

Example 1.8. Linear dependence of functions can be deduced from known identities. For example, the functions

$$f_1 = \sin(2x), f_2 = \cos(2x), and f_3 = 5$$

form a linearly dependent set in $F(\infty, \infty)$, since the equation

$$5f_1 + 5f_2 - f3 = 5\sin(2x) + 5\cos(2x)5 = 5(\sin(2x) + \cos(2x)) - 5 = 0$$

expresses 0 as a linear combination of f_1, f_2, and f_3 with coefficients that are not all zero.

In general, it is relatively rare that linear independence or dependence of functions can be ascertained by algebraic or trigonometric methods. To make matters worse, there is no general method for doing that either. That said, there does exist a theorem that can be useful for that purpose in certain cases. The following definition is needed for that theorem.

Definition 1.2. If $f_1 = f_1(x), f_2 = f_2(x), ..., f_n = f_n(x)$ are functions that are $n-1$ times differentiable on the interval $(-\infty, \infty)$, then the determinant

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) & \dots & f_n(x) \\ f_1'(x) & f_2'(x) & \dots & f_n'(x) \\ f_1''(x) & f_2''(x) & \dots & f_n''(x) \\ \vdots & \vdots & \dots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \dots & f_n^{(n-1)}(x) \end{vmatrix}$$

is called the **Wronskian** of f_1, f_2, ... , f_n.

Theorem 1.3. If the functions f_1, f_2, ... , f_n have $n-1$ continuous derivatives on the interval $(-\infty, \infty)$, and if the Wronskian of these functions is **not identically zero** on $(-\infty, \infty)$, then these functions form a **linearly independent** set of vectors in $C^{(n-1)}(-\infty, \infty)$.

Example 1.9. Use the Wronskian to show that $f_1 = x$ and $f_2 = \sin(x)$ are linearly independent vectors in $C^{(n-1)}(-\infty, \infty)$.

Example 1.10. Use the Wronskian to show that $f_1 = 1$, $f_2 = e^x$, and $f_3 = e^{2x}$ are linearly independent vectors in $C^{(n-1)}(-\infty, \infty)$.

2 Basis

Definition 2.1. A vector space V is said to be **finite-dimensional** if there is a finite set of vectors in V that spans V and
it is said to be **infinite-dimensional** if no such set exists.

Definition 2.2. If $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., v_n\}$ is a set of vectors in a finite-dimensional vector space V , then S is called a basis for V if:
(a) S spans V.
(b) S is linearly independent.

Example 2.1. Standard Basis

- The Standard Basis for $\mathbb{R}^n$:

- The Standard Basis for P_n:

- The Standard Basis for M_{22}, the set of 2×2 matrices:

Example 2.2. Showthat the vectors $v_1 = (1, 2, 1)$, $v_2 = (2, 9, 0)$, and $v_3 = (3, 3, 4)$ form a basis for $\mathbb{R}^3$.

Theorem 2.1. (Uniqueness of Basis Representation)
If $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a basis for a vector space V, then every vector $\mathbf{v}$ in V can be expressed in the form

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

in exactly one way.

Proof:

Definition 2.3. If $S = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a basis for a vector space V, and

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

is the expression for a vector $\mathbf{v}$ in terms of the basis S, then the scalars $c_1, c_2, ..., c_n$ are called the **coordinates** of $\mathbf{v}$ relative to the basis S. The vector $(c_1, c_2, ..., c_n)$ in $\mathbb{R}^n$ constructed from these coordinates is called the **coordinate vector** of $\mathbf{v}$ relative to S; it is denoted by

$$(v)_S = (c_1, c_2, ..., c_n).$$

Remarks:

- The order in which the element in each basis set are listed is critical for coordinate vectors, since changing the order of the basis vectors changes the coordinate vectors. To deal with this complication, we can define an **ordered basis** to be one in which the listing order of the basis vectors remains fixed.
- Notice that $(v)_S$ is a vector in $\mathbb{R}^n$, so that once an ordered basis S is given for a vector space V, Theorem 2.1 establishes a one-to-one correspondence between vectors in V and vectors in Rn

Example 2.3. CoordinateVectors Relative to Standard Bases

- Find the coordinate vector for the polynomial

$$p(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$$

relative to the standard basis for the vector space P_n.

- Find the coordinate vector of

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

relative to the standard basis for M_{22}.

Example 2.4. Find the vector v in $\mathbb{R}^3$ whose coordinate vector relative to S is $(v)_S = (-1, 3, 2)$.