Chapter 6
Multivariable Optimization without Constraints

6.1 Definitions of Extreme Points

Similar to the definitions of extreme points of single-variable functions, we have

Definition A point $\mathbf{x}^*$ is a ***local maximum point*** of $f{:}\boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}^n$, if

$$f\left(\mathbf{x}^*\right) \geq f\left(\mathbf{x}\right),$$

for any $\mathbf{x} \in \boldsymbol{S}$ such that $\left\|\mathbf{x} - \mathbf{x}^*\right\| < \varepsilon$ for some $\varepsilon > 0$.

Definition A point $\mathbf{x}^*$ is a ***local strict maximum point*** of $f{:}\boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}^n$, if

$$f\left(\mathbf{x}^*\right) > f\left(\mathbf{x}\right),$$

for any $\mathbf{x} \in \boldsymbol{S}$, $\mathbf{x} \neq \mathbf{x}^*$, such that $\left\|\mathbf{x} - \mathbf{x}^*\right\| < \varepsilon$ for some $\varepsilon > 0$.

Definition A point $\mathbf{x}^*$ is a ***global maximum point*** of $f{:}\boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}^n$, if

$$f\left(\mathbf{x}^*\right) \geq f\left(\mathbf{x}\right),$$

for any $\mathbf{x} \in \boldsymbol{S}$.

Definition A point $\mathbf{x}^*$ is a ***global strict maximum point*** of $f{:}\boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}^n$, if

$$f\left(\mathbf{x}^*\right) > f\left(\mathbf{x}\right),$$

for any $\mathbf{x} \in \boldsymbol{S}$, $\mathbf{x} \neq \mathbf{x}^*$.

6.2 2-Variable Optimization

A 2-variable differentiable function and its total differential are given by

$$\begin{aligned} y &= f\left(x_1, x_2\right) \\ dy &= f_1 dx_1 + f_2 dx_2. \end{aligned}$$

At extreme point $\left(x_1^*, x_2^*\right)$, minimum or maximum, for any arbitrarily small changes in x_1 and x_2, in terms of dx_1 and dx_2, there is no change in y, i.e.,

$$dy = f_1\left(x_1^*, x_2^*\right)dx_1 + f_2\left(x_1^*, x_2^*\right)dx_2 = 0.$$

6.3 First-Order Necessary Condition

From the last equation, we can conclude that

Theorem If $\left(x_1^*, x_2^*\right)$ is an extreme point of a differentiable function $f\!:\!\boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}^2$, then $f_1\left(x_1^*, x_2^*\right) = 0$ and $f_2\left(x_1^*, x_2^*\right) = 0$.

At maximum point $\left(x_1^*, x_2^*\right)$, the change of the change must be non-positive, i.e.,

$$dy^2 \le 0,$$

where for any dx_1 and dx_2, with the argument $\left(x_1^*, x_2^*\right)$ is suppressed for brevity and ligibility,

$$\begin{aligned} dy^2 &= d\left(f_1\left(x_1^*, x_2^*\right)dx_1 + f_2\left(x_1^*, x_2^*\right)dx_2\right) \\ &= \left(f_{11}dx_1 + f_{12}dx_2\right)dx_1 + \left(f_{21}dx_1 + f_{22}dx_2\right)dx_2 \\ &= f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2 \\ &= \begin{bmatrix} dx_1 & dx_2 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} \\ &= \mathbf{dx}^{\mathrm{T}}\mathbf{H}\left(x_1^*, x_2^*\right)\mathbf{dx} \le 0. \end{aligned}$$

The quadratic form $\mathbf{dx}^{\mathrm{T}}\mathbf{H}\left(x_1^*, x_2^*\right)\mathbf{dx} \le 0$ for any vector **dx** means that the symmetric matrix **H** is negative semi-definite. We have the second-order necessary condition as follows.

6.4 Second-Order Necessary Condition

Theorem If $\left(x_1^*, x_2^*\right)$ is a ***maximum point*** of a twice-differentiable function $f\colon \boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}^2$, then $\mathbf{H}\left(x_1^*, x_2^*\right)$ is ***negative semi-definite***. That means,

a) $f_{11}\left(x_1^*, x_2^*\right) \le 0, f_{22}\left(x_1^*, x_2^*\right) \le 0$, and

b) $f_{11}\left(x_1^*, x_2^*\right) f_{22}\left(x_1^*, x_2^*\right) - \left[f_{12}\left(x_1^*, x_2^*\right)\right]^2 \ge 0$.

Theorem If $\left(x_1^*, x_2^*\right)$ is a ***minimum point*** of a twice-differentiable function $f\colon \boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}^2$, then $\mathbf{H}\left(x_1^*, x_2^*\right)$ is ***positive semi-definite***. That means,

a) $f_{11}\left(x_1^*, x_2^*\right) \ge 0, f_{22}\left(x_1^*, x_2^*\right) \ge 0$, and

b) $f_{11}\left(x_1^*, x_2^*\right) f_{22}\left(x_1^*, x_2^*\right) - \left[f_{12}\left(x_1^*, x_2^*\right)\right]^2 \ge 0$.

Example *Profit Maximization*. Let the production function be given by

$$q = f\left(K, L\right),$$

each finished good is sold at *p* bahts. The price of capital and labor is fixed at p_K and p_L , respectively. The profit function is thus a function of both capital *K* and labor *L*,

$$\pi\left(K, L\right) = pf\left(K, L\right) - \left(p_K K + p_L L\right).$$

If $\left(K^*, L^*\right)$ is the profit maximizing (locally) point, by the first order necessary condition,

$$\begin{aligned} \pi_K\left(K^*, L^*\right) &= pf_K\left(K^*, L^*\right) - p_K = 0 \\ \pi_L\left(K^*, L^*\right) &= pf_L\left(K^*, L^*\right) - p_L = 0. \end{aligned}$$

Thus, at the maximum point, the ***value of the marginal product*** of each input must equal its price. Since $p_K, p_L > 0$, this is equivalent to

$$\frac{f_K\left(K^*,L^*\right)}{f_L\left(K^*,L^*\right)}=\frac{p_K}{p_L}.$$

By the second-order necessary condition,

$$\begin{aligned} pf_{KK}\left(K^*,L^*\right)&\le 0\\ pf_{LL}\left(K^*,L^*\right)&\le 0\\ p^2 f_{KK}\left(K^*,L^*\right)f_{LL}\left(K^*,L^*\right)-\left(pf_{KL}\left(K^*,L^*\right)\right)^2&\ge 0. \end{aligned}$$

6.5 Sufficient Conditions These are the conditions that, when satisfied, guarantee that a point is a maximum or minimum point of a twice-differentiable function. However, these conditions may not identify every maximum and minimum points.

Theorem For a twice-differentiable function $f{:}\boldsymbol{S}\to\boldsymbol{R}$, $\boldsymbol{S}\subseteq\boldsymbol{R}^2$, if

a) $f_1\left(x_1^*,x_2^*\right)=0, f_2\left(x_1^*,x_2^*\right)=0$, and

b) $f_{11}\left(x_1^*,x_2^*\right)<0$ and $f_{11}\left(x_1^*,x_2^*\right)f_{22}\left(x_1^*,x_2^*\right)-\left[f_{12}\left(x_1^*,x_2^*\right)\right]^2>0$,

then $\left(x_1^*,x_2^*\right)$ is a ***local maximum point***.

- Both conditions must be satisfied.
- Part (b) means the Hessian is negative definite.
- The sufficient conditions actually find a local ***strict*** maximum point.

Theorem For a twice-differentiable function $f{:}\boldsymbol{S}\to\boldsymbol{R}$, $\boldsymbol{S}\subseteq\boldsymbol{R}^2$, if

a) $f_1\left(x_1^*,x_2^*\right)=0, f_2\left(x_1^*,x_2^*\right)=0$, and

b) $f_{11}\left(x_1^*,x_2^*\right)>0$ and $f_{11}\left(x_1^*,x_2^*\right)f_{22}\left(x_1^*,x_2^*\right)-\left[f_{12}\left(x_1^*,x_2^*\right)\right]^2>0$,

then $\left(x_1^*,x_2^*\right)$ is a ***local minimum pointi***.

Example Use the sufficient conditions to find the point of maximum profit, when the production function is

$q = f(K,L) = 5K^{\frac{2}{5}}L^{\frac{1}{5}}$, and $p_K = 3$, $p_L = 4$. The output price p is assumed to be 1. The profit function is then given by

$$\pi(K,L) = 5K^{\frac{2}{5}}L^{\frac{1}{5}} - 3K - 4L.$$

The partial derivatives with respect to each input at the maximum point are given by

$$2K_0^{-\frac{3}{5}}L_0^{\frac{1}{5}} - 3 = 0$$
$$K_0^{\frac{2}{5}}L_0^{-\frac{4}{5}} - 4 = 0.$$

The ***critical point*** is $(K_0,L_0) = \left(\frac{2}{9},\frac{1}{12}\right)$. Verify that the second-order sufficient conditions

$$-\frac{6}{5}K_0^{-\frac{8}{5}}L_0^{\frac{1}{5}} < 0, \text{ and}$$
$$\left(-\frac{6}{5}K_0^{-\frac{8}{5}}L_0^{\frac{1}{5}}\right)\left(-\frac{4}{5}K_0^{\frac{2}{5}}L_0^{-\frac{9}{5}}\right) - \left(\frac{2}{5}K_0^{-\frac{3}{5}}L_0^{-\frac{4}{5}}\right)^2 > 0,$$

are satisfied. Then conclude that the point $(K_0,L_0) = \left(\frac{2}{9},\frac{1}{12}\right)$ is a local maximum point.

- If all the prices p, p_K and p_L are increased by 5%, will the point $(K_0,L_0) = \left(\frac{2}{9},\frac{1}{12}\right)$ still be a local maximum point?

HW With production function $q = f(K,L) = 5K^2L$, recalculate the maximum point.

Example Find the input levels that maximize the profit when the production function is given by

$$q = f(K,L) = 5L + K^{\frac{1}{2}}L^{\frac{3}{4}},$$

and $p = 2$, $p_K = 4$, and $p_L = 16$. The profit function is then given by

$$\pi\left(K,L\right) = 2\left(5L + K^{\frac{1}{2}}L^{\frac{3}{4}}\right) - 4K - 16L\,.$$

The partial derivatives with respect to each input at the maximum point are given by

$$K_0^{-\frac{1}{2}}L_0^{\frac{3}{4}} - 4 = 0$$

$$10 + K_0^{\frac{1}{2}}L_0^{-\frac{1}{4}} - 16 = 0.$$

The ***critical point*** is $\left(K_0,L_0\right) = \left(256,256\right)$. However, the second-order sufficient conditions are not satisfied because

$$\pi_{KK}\left(K_0,L_0\right) = -\frac{1}{2}K_0^{-\frac{3}{2}}L_0^{\frac{3}{4}} < 0$$

$$\begin{aligned}\pi_{KK}\left(K_0,L_0\right)\pi_{LL}\left(K_0,L_0\right) - \left[\pi_{KL}\left(K_0,L_0\right)\right]^2 &= \left(-\frac{1}{2}K_0^{-\frac{3}{2}}L_0^{\frac{3}{4}}\right)\left(-\frac{3}{8}K_0^{\frac{1}{2}}L_0^{-\frac{5}{4}}\right) - \left(\frac{3}{4}K_0^{-\frac{1}{2}}L_0^{-\frac{1}{4}}\right)^2 \\ &= \frac{3}{16}L_0^{-\frac{1}{2}}\left(K_0^{-\frac{1}{2}} - 3K_0^{-1}\right) \\ &= \frac{3}{16}L_0^{-\frac{1}{2}}K_0^{-\frac{1}{2}}\left(1 - 3K_0^{\frac{1}{2}}\right) < 0.\end{aligned}$$

The second inequality is positive only when $0 < K_0 < \frac{1}{9}$ and $L_0 > 0$. We cannot conclude that the critical point found is a maximum point.

HW Baldani, p. 193, #7.1 (a,b,c), 7.2 (a,b,c)

6.6 Multivariable Optimization without Constraints

With the same exposition as in the 2-variable case, we can discuss the necessary and sufficient conditions of the extreme points of functions with *n* variables as follows. A differentiable function with n variables and its differential is given by

$$y = f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$$
$$dy = \nabla f(\mathbf{x})^{\mathrm{T}} \mathbf{dx} = \sum_{j=1}^{n} f_j(\mathbf{x}) dx_j$$

At extreme point $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$, minimum or maximum, for any arbitrarily small changes in terms of **dx** there is no change in y, i.e.,

$$dy = \nabla f(\mathbf{x}^*)^{\mathrm{T}} \mathbf{dx} = \sum_{j=1}^{n} f_j(\mathbf{x}^*) dx_j = 0 .$$

6.7 First-Order Necessary Condition

From the last equation, we can conclude that

Theorem If $\mathbf{x}^*$ is an extreme point of a differentiable function $f{:}\boldsymbol{S} \rightarrow \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}^n$, then

$$\nabla f(\mathbf{x}^*) = \begin{bmatrix} f_1(\mathbf{x}^*) \\ f_2(\mathbf{x}^*) \\ \vdots \\ f_n(\mathbf{x}^*) \end{bmatrix} = \mathbf{0} .$$

At maximum point $\mathbf{x}^*$, the change of the change must be non-positive, i.e.,

$$dy^2 \le 0,$$

where for any **dx**,

$$\begin{aligned} dy^2 &= d\left(\nabla f(\mathbf{x}^*)^{\mathrm{T}} \mathbf{dx} \right) = d\left(\sum_{j=1}^{n} f_j(\mathbf{x}^*) dx_j \right) \\ &= (f_{11}dx_1 + f_{12}dx_2 + \cdots + f_{1n}dx_2)dx_1 \\ &\quad + (f_{21}dx_1 + f_{22}dx_2 + \cdots + f_{2n}dx_2)dx_2 \\ &\quad + \cdots + (f_{n1}dx_1 + f_{n2}dx_2 + \cdots + f_{nn}dx_2)dx_n \\ &= \mathbf{dx}^{\mathrm{T}} \mathbf{H}(\mathbf{x}^*) \mathbf{dx} \le 0. \end{aligned}$$

That means when $\mathbf{x}^*$ is found, moving away from it in any direction $\mathbf{dx}$ infinitestimally will not increase the value of y.

The quadratic form $\mathbf{dx}^{\mathrm{T}}\mathbf{H}\left(\mathbf{x}^*\right)\mathbf{dx} \le 0$ for any vector $\mathbf{dx}$ means that the symmetric matrix $\mathbf{H}$ is negative semi-definite. (See Simon and Blume [1994] Theorem 16.2, page 383, for the test of semi-definitenes of a square matrix). We have the second-order necessary condition as follows.

6.8 Second-Order Necessary Condition

Theorem If $\mathbf{x}^*$ is a ***maximum point*** of a twice-differentiable function $f\!:\!\boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}^n$, then $\mathbf{H}\left(\mathbf{x}^*\right)$ is ***negative semi-definite***.

Theorem If $\mathbf{x}^*$ is a ***minimum point*** of a twice-differentiable function $f\!:\!\boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}^n$, then $\mathbf{H}\left(\mathbf{x}^*\right)$ is ***positive semi-definite***.

6.9 Sufficient Conditions

These are the conditions that, when satisfied, guarantee that a point is a maximum or minimum point of a twice-differentiable function. However, these conditions may not identify every maximum and minimum points.

Theorem For a twice-differentiable function f, $f\!:\!\boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}^n$, if

a) $\nabla f\left(\mathbf{x}^*\right) = \mathbf{0}$ and

b) $\mathbf{H}\left(\mathbf{x}^*\right)$ is ***negative*** definite,

then $\mathbf{x}^*$ is a ***local maximum point***.

Theorem For a twice-differentiable function f, $f\!:\!\boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}^n$, if

a) $\nabla f\left(\mathbf{x}^*\right) = \mathbf{0}$ and

b) $\mathbf{H}\left(\mathbf{x}^*\right)$ is ***positive*** definite,

then $\mathbf{x}^*$ is a ***local minimum point***.

- Both conditions must be satisfied.

- The sufficient conditions actually find a local ***strict*** maximum (minimum) point.

6.10 Test of Definiteness of the Hessian

Definition The submatrix $\mathbf{H}_k$ of $\mathbf{H}(\mathbf{x}^*)$ is called the ***leading principal submatrix*** of order k if it is the matrix $\mathbf{H}(\mathbf{x}^*)$ with the last $n-k$ rows and columns deleted.

Definition $|\mathbf{H}_k|$ is called the ***leading principal minor*** of order k.

Theorem $\mathbf{H}(\mathbf{x}^*)$ is ***positive*** definite if $|\mathbf{H}_k| > 0$, $k = 1, \ldots, n$.

Theorem $\mathbf{H}(\mathbf{x}^*)$ is ***negative*** definite if $(-1)^k |\mathbf{H}_k| > 0$, $k = 1, \ldots, n$.

HW Baldani, p. 194, #7.4 (a,b,c,d)

6.11 Concavity, Convexity and Optimization

The sufficient conditions can only guarantee that a point is a local maximum or minimum. We need additional assumption to obtain a global extreme point. One of such assumption is the concavity or convexity of the function. This is similar to the concavity and convexity of the single-variable functions as discussed in Chapter 2.

Definition A function $f\!:\boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}^n$, is a ***concave*** (convex) function, if

$$f\left(\lambda \mathbf{x}^1 + (1-\lambda)\mathbf{x}^2\right) \underset{(\le)}{\ge} \lambda f\left(\mathbf{x}^1\right) + (1-\lambda) f\left(\mathbf{x}^2\right),$$

for any $\mathbf{x}^1, \mathbf{x}^2 \in \boldsymbol{S}$, and $0 \le \lambda \le 1$.

Definition A function $f\!:\boldsymbol{S} \to \boldsymbol{R}$, $\boldsymbol{S} \subseteq \boldsymbol{R}^n$, is a ***strictly concave*** (convex) function, if

$$f\left(\lambda\mathbf{x}^1+(1-\lambda)\mathbf{x}^2\right)\underset{(<)}{>}\lambda f\left(\mathbf{x}^1\right)+(1-\lambda)f\left(\mathbf{x}^2\right),$$

for any $\mathbf{x}^1,\mathbf{x}^2\in\boldsymbol{S}$, $\mathbf{x}^1\neq\mathbf{x}^2$, and $0<\lambda<1$.

HW Baldani, p. 194, #7.6, 7.7, 7.8

Theorem If *f* is concave, then

$$\mathbf{x}^* \text{ is local max} \Leftrightarrow \mathbf{x}^* \text{ is global max.}$$

Theorem If *f* is concave and differentiable, then

$$\nabla f\left(\mathbf{x}^*\right)=\mathbf{0}\Leftrightarrow \mathbf{x}^* \text{ is global max.}$$

Theorem If $f{:}\boldsymbol{S}\to\boldsymbol{R}$, $\boldsymbol{S}\subseteq\boldsymbol{R}^n$,is twice differentiable, then

f is concave $\Leftrightarrow\mathbf{H}(\mathbf{x})$ negative semi-definite, $\mathbf{x}\in\boldsymbol{S}$.

f is strictly concave $\Leftrightarrow\mathbf{H}(\mathbf{x})$ negative definite , $\mathbf{x}\in\boldsymbol{S}$.

Theorem If a matrix is positive (negative) definite, it is nonsingular.

6.12 Comparative Statics Analysis

Consider $y=f(\mathbf{x};\mathbf{c})$, with a vector of parameters $\mathbf{c}\in\boldsymbol{R}^k$.

Suppose that the optimal solution is found by the first-order sufficient condtion at some specific value of $\mathbf{c}_0$

$$\nabla_{\mathbf{x}}f\left(\mathbf{x}^*;\mathbf{c}_0\right)=\mathbf{0}.$$

This can be considered as a system of *n* implicit functions with *n* endogeneous variables **x** and *k* exogeneous variables **c**.
If the optimal solution is found by the sufficient conditions, $\mathbf{H}\left(\mathbf{x}^*;\mathbf{c}_0\right)$ is either positive or negative definite and thus nonsingular. Thus the Implicit Function Theorem applies because

$$\nabla^2_{\mathbf{x}}f\left(\mathbf{x}^*;\mathbf{c}_0\right)=\nabla_{\mathbf{x}}\left(\nabla_{\mathbf{x}}f\left(\mathbf{x}^*;\mathbf{c}_0\right)\right)=\mathbf{H}\left(\mathbf{x}^*;\mathbf{c}_0\right).$$

By the Implicit Function Theorem, there exists a differentiable function $\mathbf{x} = \mathbf{x}(\mathbf{c})$ and $\varepsilon > 0$ such that

a) $\nabla_{\mathbf{x}} f(\mathbf{x}(\mathbf{c});\mathbf{c}) = \mathbf{0}$, for $\|\mathbf{c} - \mathbf{c}_0\| < \varepsilon$,

b) $\mathbf{x}(\mathbf{c}_0) = \mathbf{x}^*$, and

c) the gradient

$$\begin{aligned}\nabla\mathbf{x}(\mathbf{c}_0) &= -\left[\nabla_{\mathbf{x}}\left(\nabla_{\mathbf{x}} f(\mathbf{x}^*;\mathbf{c}_0)\right)\right]^{-1}\nabla_{\mathbf{c}}\left(\nabla_{\mathbf{x}} f(\mathbf{x}^*;\mathbf{c}_0)\right)\\ &= -\left[\nabla^2_{\mathbf{x}} f(\mathbf{x}^*;\mathbf{c}_0)\right]^{-1}\nabla_{\mathbf{c}}\left(\nabla_{\mathbf{x}} f(\mathbf{x}^*;\mathbf{c}_0)\right)\\ &= -\left[\mathbf{H}(\mathbf{x}^*;\mathbf{c}_0)\right]^{-1}\nabla_{\mathbf{c}}\left(\nabla_{\mathbf{x}} f(\mathbf{x}^*;\mathbf{c}_0)\right).\end{aligned}$$

Example Suppose $y = f(x_1, x_2, x_3; c)$ is a function of three decision variables and one parameter c. By the first-order condition at $(x_1^*, x_2^*, x_3^*; c_0)$ as the system of implicit functions

$$\begin{aligned} f_1(x_1^*, x_2^*, x_3^*; c_0) &= 0\\ f_2(x_1^*, x_2^*, x_3^*; c_0) &= 0\\ f_3(x_1^*, x_2^*, x_3^*; c_0) &= 0\end{aligned}$$

There exists a function $\mathbf{x} = \mathbf{x}(c)$ such that $\nabla_{\mathbf{x}} f(\mathbf{x}(c);c) = \mathbf{0}$ for $|c - c_0| < \varepsilon$, $\mathbf{x}^* = \mathbf{x}(c_0)$ and

$$\mathbf{x}'(c_0) = -\left[\mathbf{H}(\mathbf{x}^*;c_0)\right]^{-1}\nabla_c\left(\nabla_{\mathbf{x}} f(\mathbf{x}^*;\mathbf{c}_0)\right)$$

$$\begin{bmatrix} x_1'(c_0)\\ x_2'(c_0)\\ x_3'(c_0)\end{bmatrix} = -\begin{bmatrix} f_{11} & f_{12} & f_{13}\\ f_{21} & f_{22} & f_{23}\\ f_{31} & f_{32} & f_{33}\end{bmatrix}^{-1}\begin{bmatrix} f_{1c}\\ f_{2c}\\ f_{3c}\end{bmatrix},$$

where all second-order and cross-partial derivatives are evaluated at $(x_1^*, x_2^*, x_3^*; c_0)$.

By Cramer’s Rule,

$$x_3'(c_0) = -\frac{\begin{vmatrix} f_{11} & f_{12} & f_{1c} \\ f_{21} & f_{22} & f_{2c} \\ f_{31} & f_{32} & f_{3c} \end{vmatrix}}{\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}},$$

whose sign is difficult to determine without explicit computation.

If f_1 and f_2 does not involve *c* so that $f_{1c} = f_{2c} = 0$ and then

$$x_3'(c_0) = -\frac{\begin{vmatrix} f_{11} & f_{12} & 0 \\ f_{21} & f_{22} & 0 \\ f_{31} & f_{32} & f_{3c} \end{vmatrix}}{\begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}} = -\frac{f_{3c}|\mathbf{H}_2|}{|\mathbf{H}|}.$$

Thus if $\mathbf{x}^*$ is determined to be a local maximum point by the 1st and 2nd-order sufficient conditions, $-\frac{|\mathbf{H}_2|}{|\mathbf{H}|} > 0$ and thus $sign\left[x_3'(c_0)\right] = sign\left[f_{3c}\right]$.

HW Baldani, p. 194, #7.9 (a,b,c)