Assignment 4

DUE DATE: Tuesday 9^{nd}, March 2021.

I pledge to the Honor Code and to obey all rules for taking and performing homework assignments as specified by the course instructor.

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Question 1 (50 points) Your score......................

Given the daily log returns : (R_t) can be explained by the AR(2) model as following:

$$(1 - 1.5B + 0.9B^2)R_t = 0.25 + \varepsilon_t$$

where ε_t is distributed as the Gaussian White Noise with mean (μ) = 0 and variance (σ^2) =0.25
B lag-operator

Question 1.1 (10 points) Your score......................

From the above AR(2) model, Is the model weakly stationary? Write down the reverse characteristic equation and find out the conditions to support your answer.

reverse characteristic : $x^2 - 1.5x + 0.9$

$$\lambda i = \frac{-(-1.5) \pm \sqrt{(-1.5)^2 - (4)(0.9)}}{2}$$

$$\lambda 1 = \frac{1.5 + \sqrt{2.25 - 3.6}}{2} = \overset{a}{0.75} + \overset{b}{\frac{\sqrt{-1.35}}{2}}$$

$\sim 0.5809 i$

$$= 0.75 + \frac{\sqrt{1.35}\, i}{2}$$

$$R = \sqrt{0.75^2 + 0.5809^2} = 0.8999 < 1 \rightarrow \text{weak stationarity \#}$$

Question 1.2 (10 points) Your score......................

Calculate the unconditional mean: $E(R_t)$ of R_t and the conditional mean: $E(R_t|F_{t-1})$

$$(1-1.5B+0.9B^2)R_t = 0.25+\varepsilon_t \rightarrow \begin{array}{l} R_t - 1.5R_{t-1} + 0.9R_{t-2} = 0.25 + \varepsilon_t \\ R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t \end{array}$$

unconditional mean $E[R_t]$:

$$r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + a_t$$

$$E[r_t] = \phi_0 + \phi_1 E[r_{t-1}] + \phi_2 E[r_{t-2}] + \overset{0}{E[\cancel{a_t}]}$$

$\rightarrow$ Since r_t for this case is weak stationarity $\rightarrow$ $E[r_{t-1}] = E[r_{t-2}] = E[r_t]$

$$E[r_t] = \frac{\phi_0}{1-\phi_1-\phi_2}$$

$$= \frac{0.25}{1-(1.5)-(-0.9)}$$

$$= 0.625 \#$$

conditional mean $E(R_t | F_{t-1})$:

$$r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + a_t$$

$$E[r_t|F_{t-1}] = E[\phi_0|\cdot] + E[\phi r_{t-1}|\cdot] + E[\phi_2 r_{t-2}|\cdot] + \overset{0}{E[\cancel{a_t}|\cdot]}$$

$$= \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2}$$

$$= 0.25 + 1.5 r_{t-1} - 0.9 r_{t-2}$$

Question 1.3 (10 points) | Your score......................

$$(1-1.5B+0.9B^2)R_t = 0.25+\varepsilon_t \rightarrow R_t - 1.5R_{t-1} + 0.9R_{t-2} = 0.25 + \varepsilon_t$$

$$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t$$

Find out the unconditional variance: $Var(R_t)$ of R_t and conditional variance $Var(R_t|F_{t-1})$ of R_t

unconditional variance $var(R_t)$: $E[(r_t-\mu)^2] = \phi_1^2 \overset{var(r_{t-1})}{\cancel{E[(r_{t-1}-\mu)^2]}} + \phi_2^2 \overset{var(r_{t-2})}{\cancel{E[(r_{t-2}-\mu)^2]}}$

$+ \overset{\sigma^2}{\cancel{E[a_t]^2}} + 2\phi_1\phi_2 \overset{\gamma_1}{\cancel{E[(r_{t-1}-\mu)(r_{t-2}-\mu)]}}$

$+ 2\phi_1 \cancel{E[(r_{t-1}-\mu)a_t]} + 2\phi_2 \cancel{E[(r_{t-2}-\mu)a_t]}$

$var(r_t) = \phi_1^2 var(r_{t-1}) + \phi_2^2 var(r_{t-2}) + \sigma^2 + 2\phi_1\phi_2\gamma_1$

Note: we assume $var(r_t) = var(r_{t-1}) = var(r_{t-2})$

$var(r_t) - \phi_1^2 var(r_{t-1}) - \phi_2^2 var(r_{t-2}) = \sigma^2 + 2\phi_1\phi_2\gamma_1$

$$var(r_t) = \frac{\sigma^2 + 2\phi_1\phi_2\gamma_1}{(1-\phi_1^2-\phi_2^2)}$$

$$= \cancel{\frac{0.25 + 2(1.5)(-0.9)\gamma_1}{1-(1.5)^2-(-0.9)^2}} \; -0.1214 + 1.3107\gamma_1 \; \#$$

conditional variance $Var(R_t|F_{t-1})$: $r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + a_t$

$\rightarrow$ take $var(\cdot|F_{t-1})$ both side : $var(r_t|F_{t-1}) = var(\phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + a_t | F_{t-1})$

$= \overset{0}{\cancel{var(\phi_0|F_{t-1})}} + \phi_1^2 var(r_{t-1}|F_{t-1}) + \phi_2^2 var(r_{t-2}|F_{t-1})$

$+ \overset{0}{\cancel{var(a_t|F_{t-1})}} + 2cov(\phi_1 r_{t-1}, \phi_2 r_{t-2}) + \overset{0}{\cancel{2cov(\phi r_{t-1}, a_t)}} + \overset{0}{\cancel{2cov(r_{t-2}, a_t)}}$

$var(r_t|F_{t-1}) = \phi_1^2 var(r_{t-1}|F_{t-1}) + \phi_2^2 var(r_{t-2}|F_{t-1}) + 2cov(\phi_1 r_{t-1}, \phi_2 r_{t-2})$

$= (1.5)^2 var(r_{t-1}|F_{t-1}) + (-0.9)^2 var(r_{t-2}|F_{t-1})$

$+ 2cov(\phi_1 r_{t-1}, \phi_2 r_{t-2}) \; \#$

Question 1.4 (10 points) | Your score......................

Calculate the autocorrelation: ρ_l for l=1 and 2 of R_t. Also, write down the autocorrelation: ρ_l when l $\geq$ 2.

ρ_l for $l = 1$ → autocorrelation for AR(1):

$$E[\underbrace{(r_t - \mu)(r_{t-j} - \mu)}_{\gamma_j}] = \phi_1 E[\underbrace{(r_{t-1} - \mu)(r_{t-j} - \mu)}_{\gamma_{j-1}}] + E[\cancel{a_t \cdot (r_{t-j}} - \mu)] \quad 0$$

→ divide by γ_0 to transform to autocorrelation

$$\frac{\gamma_j}{\gamma_0} = \phi_1 \frac{\gamma_{j-1}}{\gamma_0} \rightarrow \rho_j = \phi_1 \rho_{j-1}$$ and weak stationarity only when $|\phi_1| < 1$

ρ_l for $l = 2$ → autocorrelation for AR(2):

$$E[\underbrace{(r_t - \mu)(r_{t-j} - \mu)}_{\gamma_j}] = \phi_1 E[\underbrace{(r_{t-1} - \mu)(r_{t-j} - \mu)}_{\gamma_{j-1}}] + \phi_2 E[\underbrace{(r_{t-2} - \mu)(r_{t-j} - \mu)}_{\gamma_{j-2}}] + E[\cancel{a_t \cdot (r_{t-j}} - \mu)] \quad 0$$

→ divide by γ_0 to transform to autocorrelation

$$\frac{\gamma_j}{\gamma_0} = \phi_1 \frac{\gamma_{j-1}}{\gamma_0} + \phi_2 \frac{\gamma_{j-2}}{\gamma_0}$$

$$\rho_j = \phi_1 \rho_{j-1} + \phi_2 \rho_{j-2}$$

$R_t - 1.5R_{t-1} + 0.9R_{t-2} = 0.25 + \varepsilon_t$

$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t$

Question 1.5 (10 points) Your score......................

$$(1-1.5B+0.9B^2)R_t = 0.25+\varepsilon_t$$

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Given $R_{1000} = 0.01$ $R_{999} = 0.02$ $R_{998} = 0.03$ | $\varepsilon_{1000} = -0.01$ $\varepsilon_{999} = -0.02$ $\varepsilon_{998} = -0.03$ Obtain 1-step, 2-setp 95 % interval forecasts for R_t at the forecast origin t = 1000. Also the ∞-step 95 % interval forecasts for R_t. Draw these intervals.

1-Step forecast: $r_t = \phi_0 + \phi_1 r_{t-1} + \phi_2 r_{t-2} + \dots + \phi_p r_{t-p} + a_t$

$r_{h+1} = \phi_0 + \phi_1 r_h + \phi_2 r_{h-1} + \dots + \phi_p r_{h+1-p} + a_t$ and $F_h = r_h, r_{h-1} \dots$

$\rightarrow \hat{r}_h(1) = E[r_{h+1} | F_h]$

$= E[\phi_0|\cdot] + E[\phi_1 r_h|\cdot] + \dots + E[\phi_p r_{h+1-p}|\cdot] + \overset{0}{\cancel{E[a_{h+1}|\cdot]}}$

$= \phi_0 + \phi_1 r_h + \phi_2 r_{h-1} + \dots + \phi_p r_{h+1-p}$

$= 0.25 + 1.5(0.01) + (-0.9)(0.02) = 0.247$

$\rightarrow r_{h+1} - \hat{r}_h(1) = a_{h+1} \,|\, e_h(1)$

$\rightarrow var(e_h(1)|\cdot) = \cancel{var(a_{h+1}|\cdot)} = \sigma_a^2$

$\rightarrow$ estimation (interval) : $\hat{r}_h(1) \pm z_{\frac{\alpha}{2}} \sqrt{var(a_{h+1}|\cdot)} = 0.247 \pm 1.96\sigma_a$ #

2-Step forecast : $E[r_{h+2}|\cdot] = \hat{r}_h(2)$

$= \phi_0 + \phi_1 \hat{r}_h(1) + \phi_2 r_{h-1}$

$= 0.25 + 1.5(0.247) + (-0.9)(0.02) = 0.6025$

$\rightarrow r_{h+2} - \hat{r}_h(2) = a_{h+2} \,|\, e_h(2) = \phi_1 \underbrace{[r_{h+1} - \hat{r}_h(1)]}_{a_{h+1}} + a_{h+2}$

$\phi_1 a_{h+1} + a_{h+2}$

$\rightarrow var(e_h(2)|\cdot) = \phi_1^2 var(a_{h+1}|\cdot) + var(a_{h+2}|\cdot) + 2cov[\phi_1 a_{h+1}, a_{h+2})\cdot]$

$= \phi_1^2\sigma_a^2 + \sigma_a^2 \rightarrow (1+\phi_1^2)\sigma_a^2$

$= (1+1.5^2)\sigma_a^2 = 3.25\sigma_a^2$

$\rightarrow$ estimation (interval) : $\hat{r}_h(2) \pm z_{\frac{\alpha}{2}} \sqrt{var(a_{h+2}|\cdot)}$

$= 0.6025 \pm 1.96\sqrt{3.25\sigma_a^2}$

$= 0.6025 \pm 1.96(1.8028\sigma_a)$ #