



12. Autocorrelation : What happens if error terms are correlated ?

12.1 Nature of Autocorrelation

In Chapter 11 , we study the problem of heteroscedasticity. In sum, the problem of heteroscedasticity ruins the minimum variance property of estimators. In this Chapter, another problem of random disturbance term is considered. That problem is **autocorrelation** among disturbance term which violates one of the assumptions for classical linear regression model (CLRM)

The nature of autocorrelation is when there is correlation among disturbance terms or

$cov(u_i, u_j | X_i, X_j) = E(u_i, u_j) \neq 0$ where $i \neq j$

$cov(u_i, u_j | x_i, x_j) = 0$ (12.1)
NO AUTO

for time series data when the random disturbance terms are autocorrelated when the data in each period is correlated. For instance, the protest in a country that reduces the amount of export of goods and services in one month may also reduce the export of the following months. Hence, in this case, the random disturbance terms in these periods will be negative to reflect the fact that the amount of export tends to be below the mean.

For cross-sectional data, the problem of autocorrelation may occur. For example, the consumption expenditure of one family may reduce due to the great flood. Also, the flood influences other families in the same way. The consumption expenditure of these families tends to be positively correlated; hence, the random disturbance term from this set of data may also be positively correlated.

$Y_t = \beta_0 + \beta_1 X_{1t} + u_t$

t	Y_t	X_t	\hat{Y}_t	$u_t = Y_t - \hat{Y}_t$
1				u_1
2				u_2
3				u_3
4				u_4
5				
6				
n				u_n

12.1 Nature of Autocorrelation

Figure 12-1 illustrates the pattern of random disturbance term when the random disturbance term faces autocorrelation problem with the increasing trend. Contrarily, Figure 12-2 depicts the case where the random disturbance term has no obvious systematic pattern, namely no autocorrelation.

Figure 12-1: Autocorrelation among disturbance term with increasing pattern

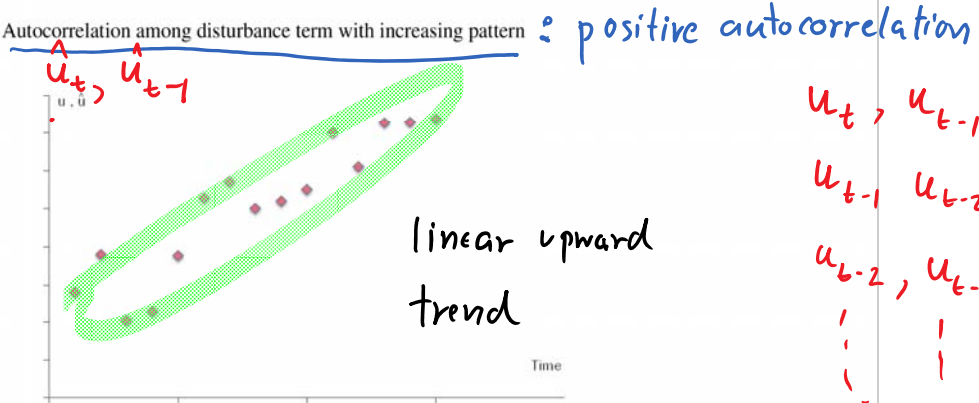


Figure 12-2: No autocorrelation among disturbance term

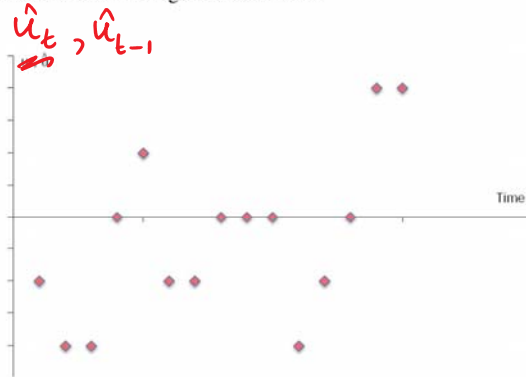
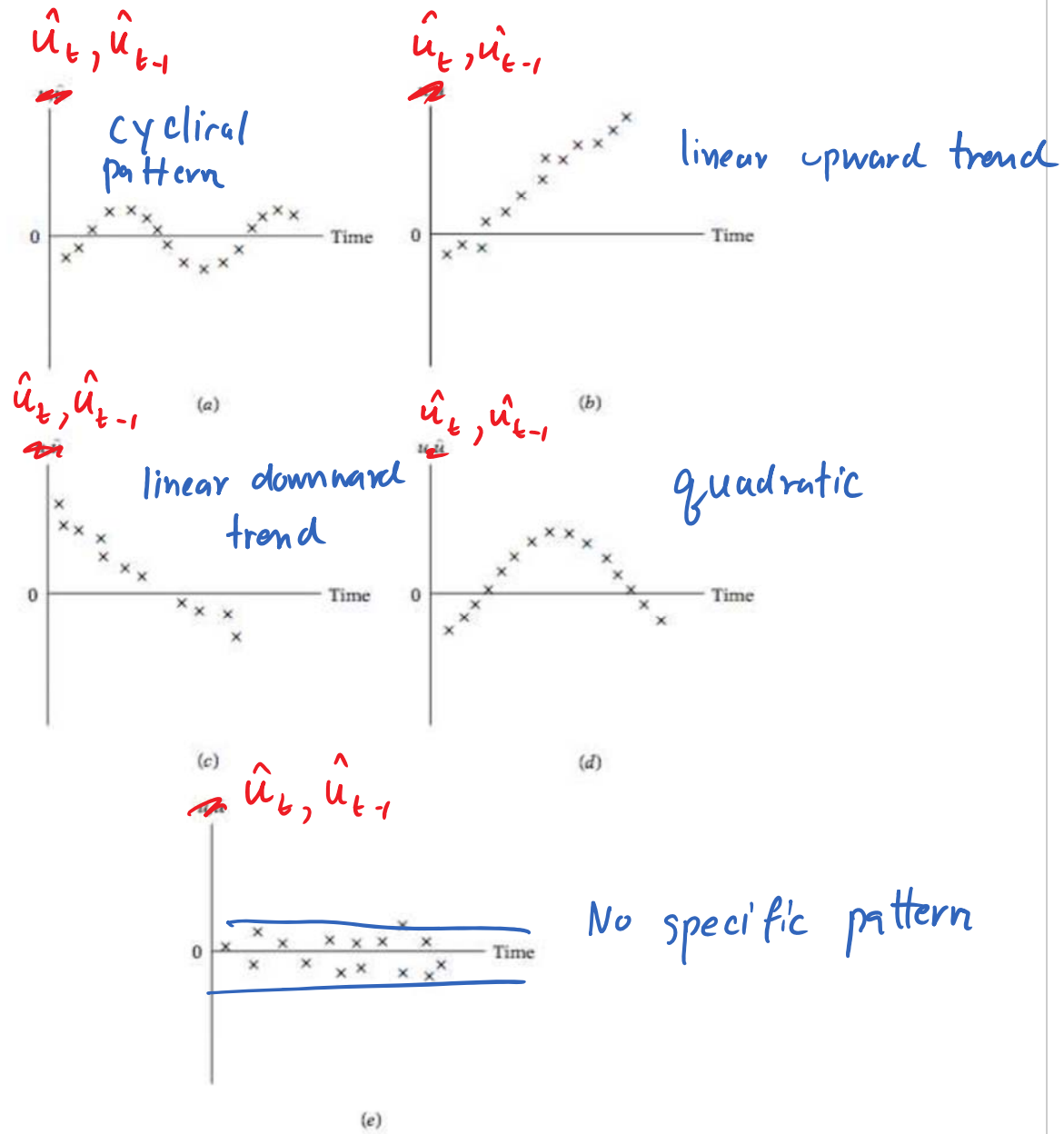


Figure 12-3: Patterns of autocorrelation and nonautocorrelation



among random disturbance terms stems from many factors. The main causes are when the model or data used in the model have the following properties.

1. The autocorrelation problem is more frequently found in the model where time series data is used than where cross-sectional one is used. The reason is that cross-sectional data involves a greater variety of observations, which tend to be independent from one another. The consumption expenditure of people in an entire country, for instance, is diverse. Any factors liable to cause an error may be negligible when the data of the entire country is employed. On the other hand, for time series data, the same sample is studied across time. Mostly, this fact results in the relationship among observations. To illustrate, macroeconomic data may indicate a positive sign in the recovery period and this trend may be prolonged until any external shock coming in.

2. **Model misspecification**, where the important regressors are omitted, could bring about the autocorrelation problem. For instance, consider the model explaining the demand for chicken with essential regressors including its price and the price of pork, as the substitute product.

$$Y_t = \beta_1 + \beta_2 X_{2t} + \beta_3 X_{3t} + u_t$$

where

correct model

Y_t = demand for chicken

X_{2t} = price of chicken

X_{3t} = price of pork

Unfortunately, suppose we wrongly specify the model such that the regressor X_{3t} is dropped and the model becomes

$$Y_t = \beta_1 + \beta_2 X_{2t} + v_t$$

$$v_t = \beta_3 X_{3t} + u_t$$

price of pork

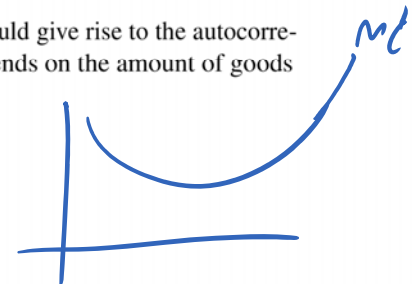
where $v_t = \beta_3 X_{3t} + u_t$. It can be seen that the random disturbance term in this misspecified model (v_t) incorporates the relationship of demand for chicken and the price of pork. This characteristic could result in significant pattern in disturbance term, leading to autocorrelation problem. The autocorrelation in this case is called **false autocorrelation** since the problem is not originated from the disturbance term itself but model misspecification instead.

3. **Model misspecification**, where the functional form is incorrect, could give rise to the autocorrelation problem as well. Consider the model of marginal cost which depends on the amount of goods produced.

$$MC_i = \beta_1 + \beta_2 \text{Output}_i + \beta_3 \text{Output}_i^2 + u_i$$

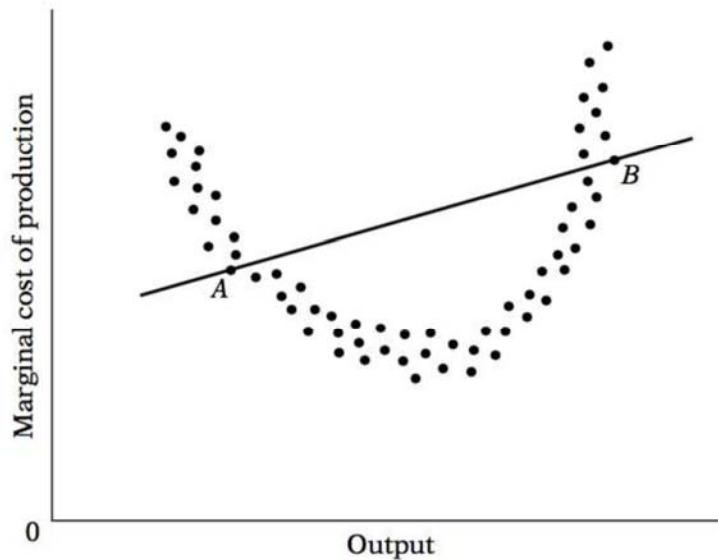
However, suppose the model is mistakenly specified as

$$MC_i = \alpha_1 + \alpha_2 \text{Output}_i + v_i$$



In this case, the random disturbance term is $v_i = \beta_3 Output_i^2 + u_i$. The result occurring is be similar to the case where the crucial regressors are neglected from the model. That is, a systematic pattern can be observed in the random disturbance term. The resulting autocorrelation is also called false autocorrelation.

Figure 12-4: Specification Bias:Incorrect Functional Form



4. **Cobweb phenomenon** might be another cause. For instance, some economists believe that supply of agricultural product displays the cobweb pattern. That is, the supplier of agricultural product makes a decision based on the last-year price as the production process takes time to deploy. The farmers have to decide first which types of plant will be produced and then production process will be carried out. Hence, they tend to base their decision on the price in the period when the type of plants is chosen rather than the price when the product is marketed.

If the price of one plant in the last period is high, there will be a great incentive for farmers to produce that plant. The product will, then, flood the market, forcing its price to go down. Contrarily, if the price of that plant in the last period is low, that plant will become unprofitable to produce in the view of farmers. This probably results in deficiency of the product, raising the price of the plant. Accordingly, the current amount of agricultural product will rely on the price last year. With the predictable pattern

$$Supply_t = \beta_1 + \beta_2 P_{t-1} + u_t$$

$$t-1 = 2560$$

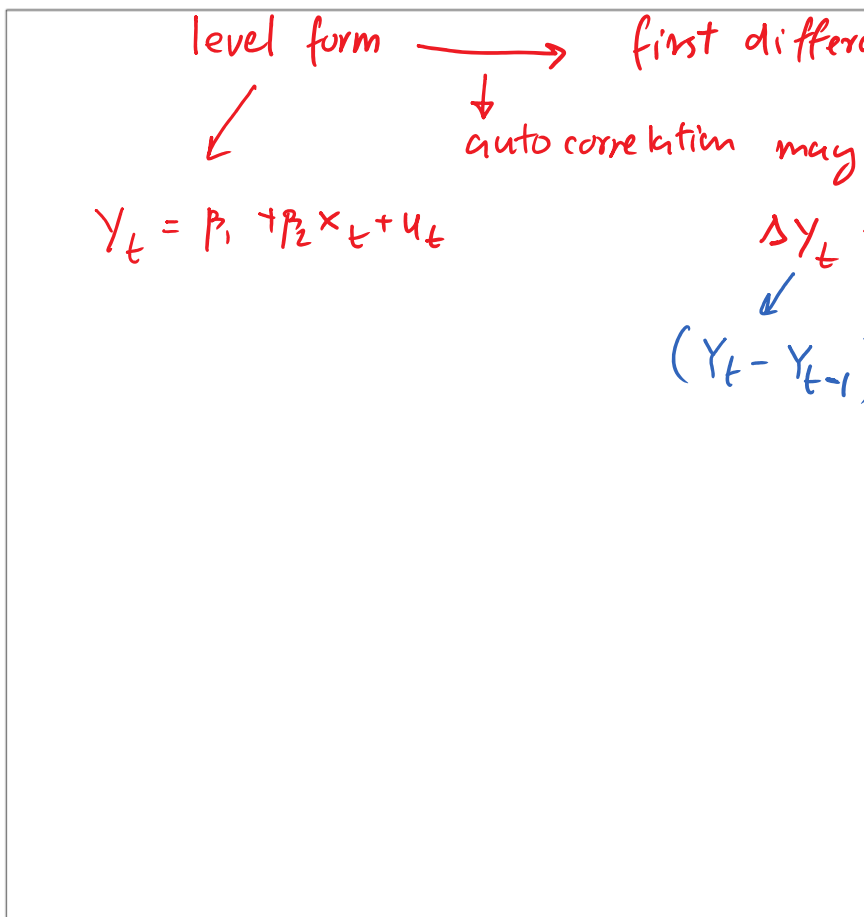
$t = 2561 \rightarrow$ at the end, if the production is over produced

$t + 1 = 2562 \rightarrow$ production will be lower

production among periods influence each other.

of regressor and regressand, the random disturbance term may display that systematic pattern, leading to the autocorrelation problem.

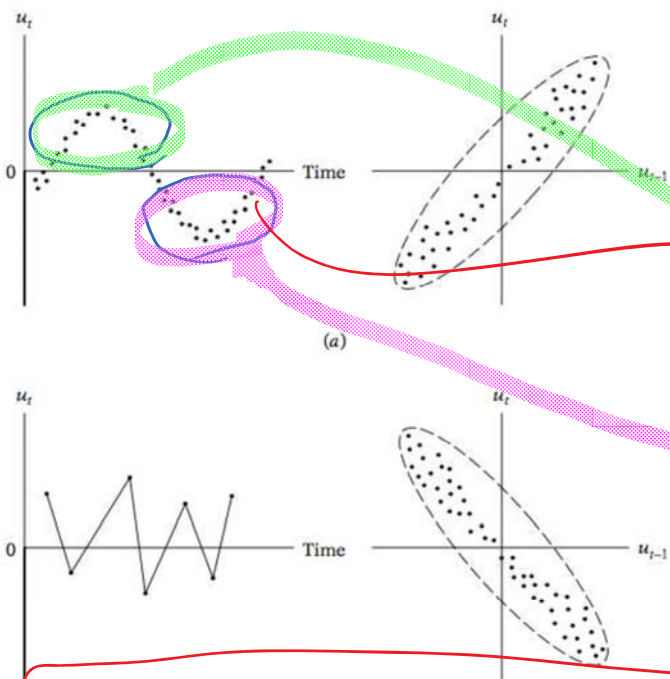
5. **The transformation of data** into first-difference form may inflict the autocorrelation problem on the model. Consider simple regression model. It is obvious that, when the model is established through first-difference transformation, autocorrelation in random disturbance term will result.



In sum, there are a variety of reasons why the error term in a regression model may be autocorrelated.

It should be noted also that autocorrelation can be positive as well as negative, although most economic time series generally exhibit positive autocorrelation because most of them either move upward or downward over extended time periods and do not exhibit a constant up-and-down movement

Figure 12-5: (a) Positive autocorrelation and (b) negative autocorrelation



When +ve errors are followed by +ve errors

and when -ve errors are followed by -ve errors

Negative errors are followed by positive errors, vice versa

12.2 Consequence of Autocorrelation

Since the autocorrelation in random disturbance term violates classical linear regression model assumptions, the following result can be shown in this section. Consider the simple linear regression model with X as explanatory variable, Y as explained variable, and the random disturbance term at time t : u_t has the relationship with the one-period-lagged disturbance term u_{t-1} .

$$Y_t = \beta_1 + \beta_2 X_t + u_t \quad (12.2)$$

$$u_t = \rho u_{t-1} + \varepsilon_t, \quad -1 < \rho < 1 \quad \rightarrow \quad \text{ARC}(1) \quad (12.3)$$

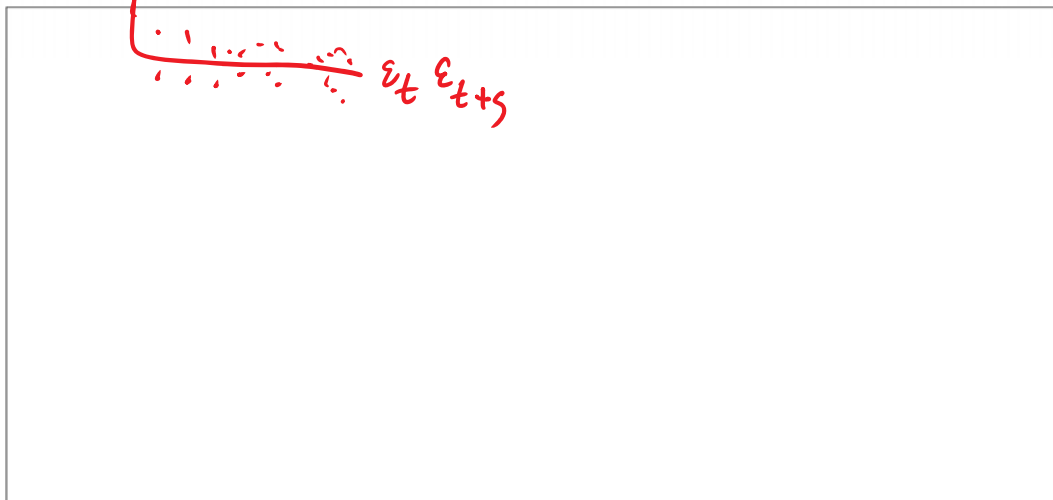
order

where ρ is the coefficient of autocovariance which specifies the degree of relationship between the disturbance term at one period and the term at lagged period. Let the value of ρ ranges between -1 and 1. According to Equation 12.3, this kind of relationship is called **first-order autoregressive: AR(1)**, namely the lag period is one. Also, if the maximum lag period is two, we call it AR(2). Generally, with the lag period of p , we can write the autoregressive model as Equation 12.4.

$$u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + \dots + \rho_p u_{t-p} + \varepsilon_t \quad \rightarrow \quad \text{ARC}(p) \quad (12.4)$$

ε is the **white noise error term** in the autoregressive model with the following properties.

$$\left. \begin{aligned} E(\varepsilon_t) &= 0 \\ \text{var}(\varepsilon_t) &= \sigma_\varepsilon^2 \\ \text{cov}(\varepsilon_t, \varepsilon_{t+s}) &= 0 \quad \text{where } s \neq 0 \end{aligned} \right\} \text{well-behaved errors } \text{😊}$$



It can be seen that, if the coefficient of autocovariance ρ is equal to 1 or -1, the variance of the error term will be undefined. We have to specify the value of ρ between this range to make the disturbance term stationary. Otherwise, the disturbance term may deviate from what it should be, namely in non-stationary disturbance term, so that the regression analysis is inapplicable.

Consider simple regression model without first-order autoregressive disturbance term.

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

Through OLS method, the estimator has the following close form solution.

$$\hat{\beta}_2 = \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sum(X_i - \bar{X})^2} = \frac{\sum x_i y_i}{\sum x_i^2}$$

The variance of the estimator can be calculated by the following formula.

$$\text{Var}(\hat{\beta}_2) = \frac{\sigma^2}{\sum(X_i - \bar{X})^2} = \frac{\sigma^2}{\sum x_i^2}$$

Nevertheless, under AR(1) scheme, the variance of the estimator can be computed as:

OLS ESTIMATORS ARE BLUE UNDER NO AUTO CORRELATION

$$\begin{aligned} \text{Var}(\hat{\beta}_2)_{AR(1)} &= \frac{\sigma^2}{\sum x_t^2} \cdot \left[1 + 2\rho \frac{\sum x_t x_{t-1}}{\sum x_t^2} + 2\rho^2 \frac{\sum x_t x_{t-2}}{\sum x_t^2} + \dots + 2\rho^{n-1} \frac{\sum x_t x_1}{\sum x_t^2} \right] \\ &= \frac{\sigma^2}{\sum x_t^2} \cdot [1 + 2\rho] = \text{Var}(\hat{\beta}_2) \cdot \left[\frac{1 + 2\rho}{1 - 2\rho} \right] \end{aligned}$$

The difference between the two above situation is that the variance under AR(1) scheme will be higher. In Equation 12.3, as ρ is equal to zero, or there is no autocorrelation in disturbance term, Equation 12.3 will converge to the usual formula of the variance of the estimator. Hence, due to autocorrelation problem, the OLS estimators will not possess best property, namely the estimator will not have minimum variance !! But the OLS estimators are still unbiased.

Ex: If $\rho = 0.6$, $\rho = 0.8$

$$\text{Var}(\hat{\beta}_2)_{AR(1)} = 2.8461 \cdot \text{Var}(\hat{\beta}_2)_{OLS}$$

$$\text{or } \text{Var}(\hat{\beta}_2)_{OLS} = \frac{1}{2.8461} \text{Var}(\hat{\beta}_2)_{AR(1)} = 0.3513 \cdot \text{Var}(\hat{\beta}_2)_{AR(1)}$$

When autocorrelation occurs and you use formula ① instead of formula ② to compute $\text{var}(\hat{\beta}_2)$, you actually "UNDERESTIMATE" the correct value of $\text{var}(\hat{\beta}_2)$!!!

• Wrong use of $\text{var}(\hat{\beta}_2) \rightarrow$ wrong $\text{se}(\hat{\beta}_2)$

↓
wrong t

↓

wrong conclusion on
hypothesis testing

- SUMMARY: No AUTO \rightarrow use formula ① : $\text{var}(\hat{\beta}_2)_{OLS}$
W/ AUTO \rightarrow use formula ② : $\text{var}(\hat{\beta}_2)_{ARCI}$
to compute $\text{var}(\hat{\beta}_2)$

12.2.1 Detection of Autocorrelation

When the autocorrelation problem leads to some undesirable properties of the estimators, the conclusion drawn from hypothesis testing may be misleading. Therefore, to prevent this mistake, we should examine whether the model suffers this problem. There are many approaches used to detect this problem and some of them are discussed in this section.

1. *Finding the relationship among disturbance terms* *by graph*: as depicted in Figure 12-3, if there is autocorrelation, the pattern of disturbance term across time will have systematic form.

2. *t-test for autocorrelation*: we can examine AR(1) model or $u_t = \rho u_{t-1} + \varepsilon_t$ and the independent variables in the model have to be **strictly exogenous**. That is,

$$E(u_t | X_{2t}, X_{3t}, \dots, X_{kt}) = 0$$

or

$$\text{cov}(u_t, X_{jt}) = 0 \text{ where } j = 2, 3, \dots, k$$

When the independent variables are strictly exogenous, we can set the following hypothesis as

$$H_0 : \rho = 0 \text{ and } H_a : \rho \neq 0$$

with the following test procedure.

Step 1: estimate the model of interest through OLS method to obtain \hat{u}_t

Step 2: construct AR(1) model with dependent variable u_t and independent variable u_{t-1} and, then, estimate the regression model to obtain the estimated value of ρ .

$$\hat{u}_t = \hat{\rho} \hat{u}_{t-1} + \hat{\varepsilon}_t$$

AR(1) process

Step 3: calculate t-statistic of the estimator $\hat{\rho}$ and test for statistical significance. If we can reject the null hypothesis, that means there is the first order autocorrelation among disturbance terms.

This approach can be applied to the test for higher order of autoregressive model like $u_t = \rho u_{t-3} + \varepsilon_t$ which includes setting hypothesis, computing t-statistic, comparing it with the critical value in statistical table, and drawing the conclusion of whether the null hypothesis should be rejected.

3 **Durbin-Watson test**: DW-statistic test is one of the popular methods used to detect first order autocorrelation problem. Six assumptions are required for DW-test.

Assumption 1: the model has to include the intercept ✓

Assumption 2: explained variable X is non stochastic ✓

Assumption 3: the relationship of the disturbance term has to be generated by AR(1) process.

Assumption 4: the disturbance term u_t is normally distributed.



NORMALITY TEST OF \hat{u}

Assumption 5: the model under examination does not include lagged regressand Y_{t-1} as the explanatory variable. That is, if the model follows equation below, we cannot apply DW-statistic to the test for autocorrelation.

$$Y_t = \beta_1 + \beta_2 X_{2t} + \alpha Y_{t-1} + u_t$$

Assumption 6: There is no missing data.

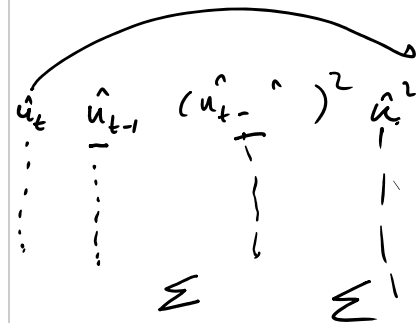
When all the assumptions are satisfied, DW-statistic can be computed by Equation 75.

$$DW = \frac{\sum_{t=2}^n (\hat{u}_t - \hat{u}_{t-1})^2}{\sum_{t=1}^n \hat{u}_t^2} \approx 2(1 - \hat{\rho}) \quad (12.5)$$

if $\hat{\rho} = 0 \rightarrow DW = 2$ ☺
 if $\hat{\rho} = 1 \rightarrow DW = 0$
 if $\hat{\rho} = -1 \rightarrow DW = 4$

$$\hat{\rho} = \frac{\sum \hat{u}_t \hat{u}_{t-1}}{\sum \hat{u}_t^2} \quad (12.6)$$

+ve -ve
 0 2 4 DW



where $\hat{\rho}$ ranges from -1 to 1, causing DW-statistic to range from 0 to 4. Consider the following possible cases. If $\hat{\rho}$ approaches 0, DW will approach 2, that is no first-order autocorrelation among disturbance terms. If $\hat{\rho}$ approach 1, DW will approach 0, that is positive first-order autocorrelation among disturbance terms. If $\hat{\rho}$ approach -1, DW will approach 4, that is negative first-order autocorrelation among disturbance terms.

After DW-statistic is obtained, we can use it to test the following hypothesis.

$$H_o : \rho = 0 \text{ and } H_a : \rho \neq 0$$

¹Durbin, James. and Watson, Geoffrey. (1951). "Testing for Serial Correlation in Least-Squares Regression," Biometrika, Vol.38 pp.159-171

The DW-statistic has to be compared with DW-statistical table invent in 1950 in order to draw the conclusion about the test. The critical value will include d_L and d_U which are the upper and lower bound respectively. The degree of freedom $k - 1$ (which is the amount of explanatory variables excluding intercept term) and level of significance (α) of 0.05 and 0.01 will vary according to the circumstance. The comparison of DW-statistic to the critical value provides two beneficial insights.

If it turns out that the estimate of the coefficient of autocovariance is greater than 0, or equivalently DW-statistic is lower than 2, it can be suspected that the random disturbance terms may have **positive autocorrelation**. Hence, the null and alternative hypothesis can be set as

$$H_0 : \rho \leq 0 \text{ and } H_a : \rho > 0$$

We cannot reject the null hypothesis when calculated DW-statistic is greater than d_U . We reject the null hypothesis when calculated DW-statistic is lower than d_L . That is, disturbance term has no positive serial correlation at the level of significance α . Nevertheless, if DW-statistic lies between d_L and d_U ($d_L \leq DW \leq d_U$), we cannot conclude whether the disturbance term has positive serial correlation.

If, on the other hand, it appears that the estimate of the coefficient of autocovariance is less than 0, or equivalently DW-statistic is greater than 2, it can be suspected that the random disturbance terms may have **negative autocorrelation**. Hence, the null and alternative hypotheses can be set as

$$H_0 : \rho \geq 0 \text{ and } H_a : \rho < 0$$

We cannot reject the null hypothesis when calculated DW-statistic is greater than $4 - d_U$. We reject the null hypothesis when calculated DW-statistic is greater than $4 - d_L$. That is, disturbance term has no positive serial correlation at the level of significance α . Nevertheless, if DW-statistic lies between $4 - d_L$ and $4 - d_U$ ($4 - d_U \leq DW \leq 4 - d_L$), we cannot conclude whether the disturbance term has negative serial correlation. Figure 12.6 illustrates the criterion whether reject or not reject the null hypothesis.

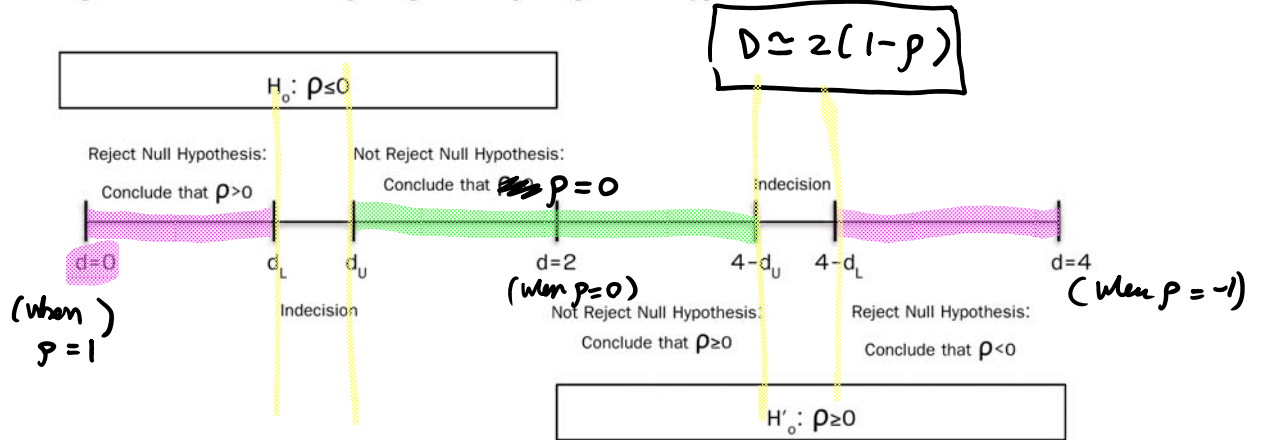
In short, the procedure to test for first order autocorrelation among disturbance terms with DW-statistic involves the following steps.

Step 1: estimate the model of interest to find \hat{u}_t

Step 2: calculate DW-statistic by the formula in Equation 12.5

Step 3: compare DW-statistic with the critical d from the table with the criterion shown in Figure 12.6 to achieve the conclusion about the relationship among disturbance terms.

Figure 12.6: Criterion for rejecting or not rejecting the null hypothesis with DW-statistic



4. **Breusch-Godfrey test**²: this method can be used to test for any order, from 1 or AR(1) to p or AR(p) of autocorrelation as illustrated in Equation 12.4 from the simple regression model

$$u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + \dots + \rho_p u_{t-p} + \varepsilon_t \tag{12.4}$$

We can form the hypothesis about the relationship among these disturbance terms as
 $H_0 : \rho_1 = \rho_2 = \dots = \rho_p = 0$
 $H_a : \text{otherwise}$
 with the following procedure.

Step 1: establish the regression model of interest. The example shown is the simple one.

Step 2: establish another model to obtain the relationship among disturbance terms with \hat{u}_t as the explained variable and X_{2t} and $\hat{u}_{t-1}, \hat{u}_{t-2}$ until \hat{u}_{t-p} as the explanatory variables.

$$\hat{u}_t = \alpha_1 + \alpha_2 X_{2t} + \hat{\rho}_1 \hat{u}_{t-1} + \hat{\rho}_2 \hat{u}_{t-2} + \dots + \hat{\rho}_p \hat{u}_{t-p} + \varepsilon_t \tag{12.7}$$

Step 3: if the sample size is large, LM-statistic can be computed by

$$LM = (n - p)R^2 \sim \chi_p^2 \tag{12.8}$$

Step 4: compare LM-statistic with the critical value of chi-square table to conclude whether the null hypothesis should be rejected. Specifically, we reject the null hypothesis if $(n - p)R^2$ is greater than the critical chi-square at the chosen level of significance and conclude that there exists the autocorrelation problem.

²Breusch, Trevor .S. (1978) "Testing for Autocorrelation in Dynamic Linear Models", Australian Economic Papers, Vol.17, Issue.31, pp.334-355. Godfrey, Leslie G. (1978). "Testing Against General Autoregressive and Moving Average Error Models When the Regressor Includes Lagged Dependent Variables," Econometrica, Vol.46, No.6, pp.1293-1301.

12.2.2 Remedial Measure for Autocorrelation

After the problem is detected, to prevent the problem from making the variance of estimators so unreliable that the conclusion drawn from hypothesis test is misleading, the remedial measure is necessary. In this section, only the remedial measures for the first order autocorrelation are discussed.

Case when the ρ is known

1. *Generalized least squares (GLS)*: consider the simple regression model with the first order autocorrelation problem AR(1)

$$Y_t = \beta_1 + \beta_2 X_t + u_t \quad \text{--- (1)}$$

$$u_t = \rho u_{t-1} + \varepsilon_t$$

We can separate the procedure into the case where the coefficient of autocovariance (ρ) is known and unknown. For the case when the ρ is **known**, we can solve the problem by

$$\begin{aligned} Y_{t-1} &= \beta_1 + \beta_2 X_{t-1} + u_{t-1} \quad \text{---} \\ \rho Y_{t-1} &= \rho \beta_1 + \rho \beta_2 X_{t-1} + \rho u_{t-1} \quad \text{---} \end{aligned}$$

$$\begin{aligned} (Y_t - \rho Y_{t-1}) &= \beta_1(1 - \rho) + \beta_2(X_t - \rho X_{t-1}) + (u_t - \rho u_{t-1}) & \textcircled{4} &= \textcircled{1} - \textcircled{3} \\ (Y_t - \rho Y_{t-1}) &= \beta_1(1 - \rho) + \beta_2(X_t - \rho X_{t-1}) + \varepsilon_t \end{aligned}$$

$$Y_t^* = \beta_1^* + \beta_2^* X_t^* + \varepsilon_t$$

where

$$\begin{aligned} \varepsilon_t &= u_t - \rho u_{t-1} \\ Y_t^* &= Y_t - \rho Y_{t-1} \\ X_t^* &= X_t - \rho X_{t-1} \\ \beta_1^* &= \beta_1(1 - \rho) \end{aligned}$$

According to the above property of ε_t , we find that the classical linear regression model assumption is satisfied. Hence, it can be concluded that, the estimators in the new model generated from the above procedure are best linear unbiased estimators (BLUE).

Notwithstanding, due to the above procedure, the regressor and regressand of the new model is in the difference form which means that one observation is lost. Thus, to mitigate the problem, the first observation on X and Y may be transformed to $X_1^* = X_1 \sqrt{1 - \rho^2}$ and $Y_1^* = Y_1 \sqrt{1 - \rho^2}$. We call this transformation process **Prais-Winsten transformation**.

Case when the ρ is unknown

In the case when the ρ is **unknown**, two solutions to the problem are available. The first one is *first difference method*. This approach is usually used when DW-statistic for the model of interest is less than R^2 . The first-difference model is constructed as:

$$\begin{aligned} Y_t - Y_{t-1} &= \beta_1 - \beta_1 + \beta_2(X_t - X_{t-1}) + (u_t - u_{t-1}) \\ \Delta Y &= \beta_2 \Delta X_t + \varepsilon_t \end{aligned}$$

When the above regressor and regressand are obtained, we can apply the OLS method for non-intercept model to estimate this new model which solves the first order autocorrelation problem.

The other method is to use estimator $\hat{\rho}$ obtained from the calculation of DW-statistic. From Equation 75 and 76, when the sample size is large enough, we can compute the value of $\hat{\rho}$ through equation 79.

$$\hat{\rho} = 1 - \frac{d}{2} \tag{12.9}$$

After we obtain the estimate, we can apply it to the remedial measure stated above when the value of ρ is known.

2. Heteroscedasticity and autocorrelation-consistent standard error: when the sample size is large enough, Newey and West suggest the formula for the variance of estimators when the autocorrelation and heteroscedasticity problem occur. With this formula, the standard deviation required in statistical analysis, such as hypothesis test, confidence interval and so forth, will be applicable.

At the present time, most econometric computer packages provide this estimation of variance in the set of statistical results. Although this method does not lessen the degree of autocorrelation problem, the obtained standard error is fixed and applicable for further statistical analysis.

It is noteworthy that the difference between the method of White and Newey-West is recognized. The approach suggested by White can solve the specific problem of heteroscedasticity; whereas the one suggested by Newey-West is designed to tackle the problems of both heteroscedasticity and autocorrelation.