OLS-GLS

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$$\mathrm{Var}(u_i) = E\big[(u_i - \overset{=0}{E(u_i)})^2\big] = E[u_i^2] = \sigma^2$$

$$E(u_i^2) = \sigma_i^2 \Rightarrow \text{Heteroscedasticity}$$

No Autocorrelation

$$\mathrm{cov}(u_t u_s) = E\big[(u_t - \overset{=0}{E(u_t)})(u_s - \overset{=0}{E(u_s)})\big] = E[u_t \cdot u_s] = 0 \quad \forall\, t \neq s$$

$$E(u_t u_{t-1}) = \rho \Rightarrow \text{First Order Autocorrelation}$$

Parameters.

$$Y_i \sim N\big(\, \underbrace{\beta_1 + \beta_2 X_{2i} + \cdots + \beta_k X_{ki}}_{} \,\big) + u_i \sim N(0, \sigma^2)$$

Loss DF.

Fixed.

$$Y_i = \beta_1 + \beta_2 X_{2i} + u_i \rightarrow \hat{\beta}_2 = \frac{\sum (X_{2i} - \bar{X}_2)(Y_i - \bar{Y})}{\sum (X_{2i} - \bar{X}_2)^2}$$

OLS Matrix Approach

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \cdots + \beta_k X_{ki} + u_i \quad \forall\, i = 1, 2, \ldots, n$$

$$\begin{array}{l} i=1 \\ i=2 \\ \vdots \\ i=n \end{array}
\underbrace{\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}}_{n\times 1}
\begin{array}{c} = \\ = \\ \\ = \end{array}
\underbrace{\begin{bmatrix} \beta_1 + \beta_2 X_{21} + \beta_3 X_{31} + \cdots + \beta_k X_{k1} \\ \beta_1 + \beta_2 X_{22} + \beta_3 X_{32} + \cdots + \beta_k X_{k2} \\ \vdots \\ \beta_1 + \beta_2 X_{2n} + \beta_3 X_{3n} + \cdots + \beta_k X_{kn} \end{bmatrix}}_{n\times 1}
+ \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}}_{n\times 1}$$

$$\underbrace{\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}}_{n\times 1}
= \underbrace{\begin{bmatrix} 1 & X_{21} & X_{31} & \cdots & X_{k1} \\ 1 & X_{22} & X_{32} & \cdots & X_{k2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_{2n} & X_{3n} & \cdots & X_{kn} \end{bmatrix}}_{n\times k}
\cdot \underbrace{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_k \end{bmatrix}}_{k\times 1}
+ \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}}_{n\times 1}$$

$$\underset{n\times 1}{Y} = \underset{n\times k}{X}\,\underset{k\times 1}{\beta} + \underset{n\times 1}{u}$$

$$\hat{\beta}$$

$$Y = X\hat{\beta} + \hat{u}$$

$$\underset{n\times 1}{\hat{u}} = Y - X\hat{\beta}$$

$$\underset{n\times 1}{\hat{u}} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_n \end{bmatrix} \qquad \underset{1\times n}{\hat{u}'} = \begin{bmatrix} \hat{u}_1 & \hat{u}_2 & \cdots & \hat{u}_n \end{bmatrix}$$

$$\underset{\substack{1\times n \;\; n\times 1 \\ (1\times 1)}}{\hat{u}'\hat{u}} = \hat{u}_1\hat{u}_1 + \hat{u}_2\hat{u}_2 + \cdots + \hat{u}_n\hat{u}_n = \hat{u}_1^2 + \hat{u}_2^2 + \cdots + \hat{u}_n^2 = \sum_{i=1}^{n} \hat{u}_i^2$$

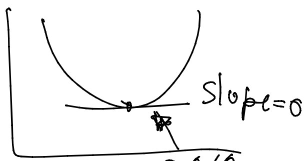
$$\text{OLS} \rightarrow \overset{\text{Min.}}{\boxed{\sum_{i=1}^{n} \hat{u}_i^2 = 0}}$$

$$\left.\begin{aligned} \frac{\partial \sum \hat{u}_i^2}{\partial \hat{\beta}_1} &= 0 \\ &= 0 \\ &\vdots \\ \frac{\partial \sum \hat{u}_i^2}{\partial \hat{\beta}_k} &= 0 \end{aligned}\right\} \begin{array}{l} k\text{-eq.} \\ k\text{-unknown} \end{array}$$

$$\hat{u}'\hat{u} = (Y - X\hat{\beta})'(Y - X\hat{\beta}) = Y'Y - \underset{\substack{1\times n \;\; n\times k \;\; k\times 1 \\ (1\times 1)}}{Y'X\hat{\beta}} - \underset{\substack{1\times k \;\; k\times n \;\; n\times 1 \\ (1\times 1)}}{\hat{\beta}'X'Y} + \hat{\beta}'X'X\hat{\beta}$$

$$\hat{u}'\hat{u} = Y'Y - 2\hat{\beta}'X'Y + \hat{\beta}'X'X\hat{\beta}$$

$$\frac{\partial \hat{u}'\hat{u}}{\partial \hat{\beta}} = 0 - 2X'Y + 2X'X\hat{\beta} = 0$$



$$\frac{\partial \hat{u}'\hat{u}}{\partial \hat{\beta}} = 0$$

$$\cancel{2}X'X\hat{\beta} = \cancel{2}X'Y$$

$$\cancel{(X'X)^{-1}X'X}\,\hat{\beta} = (X'X)^{-1}X'Y$$

$$\underset{k\times 1}{\hat{\beta}_{OLS}} = \underset{\substack{k\times n \;\; n\times k \;\; k\times n \;\; n\times 1 \\ (k\times 1)}}{(X'X)^{-1}X'Y}$$

$$\underset{(n\times 1)}{u} = \begin{bmatrix} \cancel{u_1} \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \underset{(1\times n)}{u'} = \begin{bmatrix} \cancel{u_1} & \cancel{u_2} & \cdots & u_n \end{bmatrix}$$

$$E(\underset{n\times 1}{u} \cdot \underset{1\times n}{u'}) = \begin{bmatrix} E(u_1u_1) & E(u_1u_2) & \cdots & E(u_1u_n) \\ E(u_2u_1) & E(u_2u_2) & \cdots & E(u_2u_n) \\ \vdots & & & \\ E(u_nu_1) & E(u_nu_2) & \cdots & E(u_nu_n) \end{bmatrix}$$

$E(X) = \text{Mean}$

$E[(X-E(X))^2] = \text{variance}$

$E[(X-E(X))(Y-E(Y))] = \text{covariance}$

$E(u_i) = 0$

$E[(u_i - \underbrace{E(u_i)}_{=0})^2] = E[u_i^2] = \text{variance} = \sigma_i^2$

$E[(u_i - \underbrace{E(u_i)}_{=0})(u_j - \underbrace{E(u_j)}_{=0})] = \text{cov.}$

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix}$$

$E[u_iu_j] = \text{cov}(u_iu_j) = \sigma_{ij}$

$$\underset{k\times 1}{\hat{\beta}_{GLS}} = (\underset{k\times n}{X'}\,\underset{n\times n}{\Sigma^{-1}}\,\underset{n\times k}{X})^{-1}\,\underset{k\times n}{X'}\,\underset{n\times n}{\Sigma^{-1}}\,\underset{n\times 1}{Y}$$

OLS assume:

$\Sigma = \sigma^2 I$

$$\hat{\beta}_{OLS} = (X'(\sigma^2 I)^{-1}X)^{-1}X'(\sigma^2 I)^{-1}Y$$

$$= \cancel{\sigma^2}\left(\frac{1}{\cancel{\sigma^2}}\right)(X'I^{-1}X)^{-1}X'I^{-1}Y$$

$$= (X'X)^{-1}X'Y$$

OLS is special case of GLS when $\boxed{\Sigma = \sigma^2 I}$

If we have heteroscadasticity or autocorrelation Problem, we shouldn't use OLS since $\Sigma \neq \sigma^2 I$

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Then, we should use GLS

$$\hat{\beta}_{GLS} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y$$

Problem : We can't observe Σ_u^{-1}

We need to estimate $\hat{\Sigma}$

$$\hat{\beta}_{FGLS} = (X'\hat{\Sigma}^{-1}X)^{-1}X'\hat{\Sigma}^{-1}Y$$