Relations & Functions I

TU152: Fundamental Mathematics

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Introduction

Definition (Ordered Pair)

Let A be a given set.
An **ordered pair** (a, b) of elements in A is defined to be the set $\{a, \{a, b\}\}$.

- a is called the *first component* or the *first element*.
- b is called the *second component* or the *second element*.

Two ordered pairs (a, b) and (c, d) are equal if, and only if, $a = c$ and $b = d$.
Symbolically:

$$(a, b) = (c, d) \text{ means that } a = c \text{ and } b = d.$$

Example:

- Is $(3, 1) = (1, 3)$?
- Is $(3, \frac{5}{10}) = (\sqrt{9}, \frac{1}{2})$?

Introduction

Definition (Cartesian Product)

Let A and B be two given sets.
The **Cartesian product** of A and B, denoted $A \times B$ (and read "A cross B")
is the set of all ordered pairs (a, b), where a is in A and b is in B. Symbolically:

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}.$$

Example:

1. Let $A = \{1, 2, 3\}$ and $B = \{u, v\}$.
   1. Find $B \times B$.
   2. Find $A \times B$.
   3. Find $B \times A$.
   4. Is $A \times B = B \times A$?
   5. How many elements are in $A \times B$, $B \times A$, and$B \times B$?
2. Let $\mathbb{R}$ denote the set of all real numbers.
   Describe $\mathbb{R} \times \mathbb{R}$ (this is also called **Cartesian plane**).

Relations

There are many kinds of relationships in the world, such as, relationship between people who work for the same company, between parents and children, and between people who live in the same country.
In mathematics, the objects also can be related in various ways, e.g.:

- A set A may be said to be related to a set B if A is a subset of B, or if A is not a subset of B.
- A number x may be said to be "related" to a number y
  - if $x < y$,
  - if x is a factor of y, or
  - if $x^2 + y^2 = 1$.

Definition

Let A and B be sets. A relation R from A to B is a subset of $A \times B$. Given an ordered pair (x, y) in $A \times B$, x is related to y by R, written

$$x \; R \; y,$$

if, and only if, (x, y) is in R.
The set A is called the **domain** of R and the set B is called its **co-domain**.

- The notation for a relation R may be written symbolically as follows:

$$x \; R \; y \text{ means that } (x, y) \in R.$$

- The notation $x \not R y$ means that x is not related to y by R:

$$x \not R y \text{ means that } (x, y) \notin R.$$

Example: Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ and define a relation R from A to B as follows: Given any $(x, y) \in A \times B$,

$$(x, y) \in R \text{ means that } \frac{x - y}{2} \text{ is an integer.}$$

1. State explicitly which ordered pairs are in $A \times B$ and which are in R.
2. Is $1\ R\ 3$? Is $2\ R\ 3$? Is $2\ R\ 2$?
3. What are the domain and co-domain of R?

Example: Define a relation C from $\mathbb{R}$ to $\mathbb{R}$ as follows: For any $(x, y) \in \mathbb{R} \times \mathbb{R}$,

$$(x, y) \in C \text{ means that } x^2 + y^2 = 1.$$

1. Is $(1, 0) \in C$? Is $(0, 0) \in C$? Is $(-1/2, \sqrt{3}/2) \in C$?
   Is $-2\ C\ 0$? Is $0\ C\ (-1)$? Is $1\ C\ 1$?
2. What are the domain and co-domain of C?
3. Draw a graph for C by plotting the points of C in the Cartesian plane.

Arrow Diagram of a Relation

Arrow Diagram of a Relation

Suppose R is a relation from a set A to a set B. The arrow diagram for R is obtained as follows:

1. Represent the elements of A as points in one region and the elements of B as points in another region.
2. For each x in A and y in B, draw an arrow from x to y if, and only if, x is related to y by R. Symbolically:

   Draw an arrow from x to y if, and only if, $x \; R \; y$ or $(x, y) \in R$.

Example: Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$ and define relations S and T from A to B as follows: For all $(x, y) \in A \times B$,

- $$(x, y) \in S \text{ means that } x < y$$
- $$T = \{(2, 1), (2, 5)\}.$$

Draw arrow diagrams for S and T.

Functions

A function is a special case of a relation.

- A **function** f from a set A to a set B is a relation from A to B such that for every $x \in A$ there is a unique $y \in B$ such that $(x, y) \in f$.
- For $(x, y) \in f$ we use the notation $y = f(x)$.
- We call y the **image** of x under f.
- The set A is called the **domain** of f whereas B is called the **co-domain**.
- The collection of all images of f is called the **range** of f.

Definition (Functions)

A function f from a set A to a set B is a relation with domain A and co-domain B that satisfies the following two properties:

1. For every element x in A, there is an element y in B such that $(x, y) \in f$.
2. For all elements x in A and y and z in B, if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

Note: Equivalently,

1. Every element of A is the first element(component) of an ordered pair of f.
2. No two distinct ordered pairs in f have the same first element(component).

Example:

1. Show that the relation $f = \{(1, a), (2, b), (3, a)\}$ defines a function from $A = \{1, 2, 3\}$ to $B = \{a, b, c\}$. Find its range.
2. Show that the relation $f = \{(1, a), (2, b), (3, c), (1, b)\}$ does not define a function from $A = \{1, 2, 3\}$ to $B = \{a, b, c\}$.

Example:

Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$. Which of the relations R and S defined below are functions from A to B ?

1. $R = \{(2, 5), (4, 1), (4, 3), (6, 5)\}$
2. For all $(x, y) \in A \times B, (x, y) \in S$ means that $y = x + 1$.

Example:

Graph the functions $f(x) = \lfloor x \rfloor$ and $g(x) = \lceil x \rceil$on the closed interval $[-4, 4]$.