

Macroeconomics

Lecture 7.2

The Term Structure of Interest Rate

- What is the relationship between the level of an equilibrium risk free interest rate and its term to maturity?

The Term Structure of Interest Rate

- Suppose that there are markets in one-period and two-period perfectly safe loans.
- R_{1t} and R_{2t} are risk free gross rate of return from the one- and two-period safe loans, respectively. R_{1t} and R_{2t} are known with certainty at time t .
- $\therefore R_{1t}^{-1}$ is the price of a perfectly sure claim to 1 unit of consumption at time $t+1$.

$$\text{Max} \quad E_0 \sum_{t=0}^{\infty} \beta^t U(c_t),$$

$$\text{s.t.} \quad c_t + L_{1t} + L_{2t} \leq d_t + L_{1t-1}R_{1t-1} + L_{2t-2}R_{2t-2},$$

where L_{jt} is the amount lent for j periods at time t .

The Lagrangian function is,

$$J = E_0 \sum_{t=0}^{\infty} \beta^t \left[U(c_t) + \lambda_t (d_t + L_{1t-1}R_{1t-1} + L_{2t-2}R_{2t-2} - c_t - L_{1t} - L_{2t}) \right]$$

The first-order necessary conditions are

$$\frac{\partial J}{\partial c_t} = 0, \Rightarrow \quad \left[\partial U(c_t) / \partial c_t \right] - \lambda_t = 0,$$

$$\frac{\partial J}{\partial L_{1t}} = 0, \Rightarrow \quad -\lambda_t + \beta E_t \lambda_{t+1} R_{1t} = 0,$$

$$\frac{\partial J}{\partial L_{2t}} = 0, \Rightarrow \quad -\lambda_t + \beta^2 E_t \lambda_{t+2} R_{2t} = 0,$$

Substituting the first into each of the second and third conditions gives

$$E_t \left[\beta \frac{(\partial U(c_{t+1})/\partial c_{t+1})}{(\partial U(c_t)/\partial c_t)} R_{1t} \right] = 1, \quad (1)$$

$$E_t \left[\beta^2 \frac{(\partial U(c_{t+2})/\partial c_{t+2})}{(\partial U(c_t)/\partial c_t)} R_{2t} \right] = 1. \quad (2)$$

Since R_{1t} and R_{2t} are known with certainty at the beginning of t ,

$$\therefore R_{1t}^{-1} = E_t \beta \frac{(\partial U(c_{t+1})/\partial c_{t+1})}{(\partial U(c_t)/\partial c_t)} = \text{the price of a perfectly sure claim to}$$

1 unit of consumption at time $t+1$.

This price can be easily obtained from the following reasoning,

Since R_{jt} units of consumption at $t+1 = 1$ unit of consumption at time t

hence, $\mathbf{1} = R_t^{-1} \mathbf{1}$

$$R_{2t}^{-1} = E_t \beta^2 \frac{(\partial U(c_{t+2})/\partial c_{t+2})}{(\partial U(c_t)/\partial c_t)} = \text{the price of a perfectly sure claim to}$$

1 unit of consumption at time $t + 2$.

Let $U(c) = \ln c$ and set $c_t = d_t$, Eq (1) and (2) become

$$R_{1t}^{-1} = \beta E_t(d_t/d_{t+1})$$

$$R_{2t}^{-1} = \beta^2 E_t(d_t/d_{t+2})$$

Suppose that $d_{t+1} = \rho d_t \theta_{t+1}$, $\rho > 0$,

where θ_{t+1} is a sequence of i.i.d. random variables that are positive with probability one. Then

$$R_{1t}^{-1} = \frac{\beta}{\rho} \left[E\left(\frac{1}{\theta}\right) \right], \quad \text{or} \quad R_{1t} = \frac{\rho}{\beta E(\theta^{-1})},$$

$$R_{2t}^{-1} = \left(\frac{\beta}{\rho}\right)^2 \left[E\left(\frac{1}{\theta}\right) \right]^2, \quad \text{or} \quad R_{2t} = \left(\frac{\rho}{\beta E(\theta^{-1})} \right)^2.$$

The level of interest rates rises (or falls) with the term to maturity if

$$\left[\frac{\rho}{\beta E(\theta^{-1})} \right] > 1, \quad \left(\text{or} \quad \left[\frac{\rho}{\beta E(\theta^{-1})} \right] < 1 \right).$$

If $R_{1t} = (1+r) = \frac{\rho}{\beta E(\theta^{-1})} > 1$, then $r > 0$.

It also follows that $R_{2t} = \left(\frac{\rho}{\beta E(\theta^{-1})} \right)^2 = (1+r)^2 > R_{1t}$,

(The level of interest rates rises with the term to maturity)

R_{1t} and R_{2t} are risk free gross rates of return because they are the rates of return of perfectly safe loans.

The values of R_{1t} and R_{2t} are obtained from the equilibrium of the economy with uncertain future dividends.

Let us write Eq(1) as

$$\beta \frac{(\partial U(c_{t+1})/\partial c_{t+1})}{(\partial U(c_t)/\partial c_t)} R_{1t} = 1 + \varepsilon_{1t+1},$$

where $E_t \varepsilon_{1t+1}$ is the least squares residual and $E_t \varepsilon_{1t+1} = 0$.

$$\therefore \beta \frac{(\partial U(c_{t+1})/\partial c_{t+1})}{(\partial U(c_t)/\partial c_t)} = R_{1t}^{-1} + R_{1t}^{-1} \varepsilon_{1t+1} \quad (3)$$

Eq (2) can be written as

$$R_{2t}^{-1} = E_t \left[\beta^2 \frac{(\partial U(c_{t+1})/\partial c_{t+1})}{(\partial U(c_t)/\partial c_t)} \frac{(\partial U(c_{t+2})/\partial c_{t+2})}{(\partial U(c_{t+1})/\partial c_{t+1})} \right] \quad (4)$$

Substituting Eq (3) into Eq (4) gives

$$\begin{aligned} R_{2t}^{-1} &= E_t \left[(R_{1t}^{-1} + R_{1t}^{-1} \varepsilon_{1t+1}) (R_{1t+1}^{-1} + R_{1t+1}^{-1} \varepsilon_{1t+2}) \right] \\ &= R_{1t}^{-1} E_t \left[R_{1t+1}^{-1} + R_{1t+1}^{-1} \varepsilon_{1t+1} + R_{1t+1}^{-1} \varepsilon_{1t+1} \varepsilon_{1t+2} + R_{1t+1}^{-1} \varepsilon_{1t+2} \right]. \end{aligned}$$

$$\begin{aligned} \text{Or, } R_{2t}^{-1} &= R_{1t}^{-1} \left[E_t R_{1t+1}^{-1} + E_t R_{1,t+1}^{-1} E \varepsilon_{1t+1} + \text{cov}_t (R_{1t+1}^{-1}, \varepsilon_{1t+1}) \right] \\ &\quad + R_{1t}^{-1} E_t R_{1t+1}^{-1} \varepsilon_{1t+1} \varepsilon_{1t+2} + R_{1t}^{-1} E_t R_{1t+1}^{-1} \varepsilon_{1t+2} \quad (5) \end{aligned}$$

Consider the last two terms

$$\begin{aligned}
 E_t R_{1t+1}^{-1} \varepsilon_{1t+1} \varepsilon_{1t+2} &= E_t E_{t+1} R_{1t+1}^{-1} \varepsilon_{1t+1} \varepsilon_{1t+2} \\
 &= E_t R_{1t+1}^{-1} E_{t+1} \varepsilon_{1t+1} \varepsilon_{1t+2} \\
 &= E_t R_{1t+1}^{-1} E_{t+1} \varepsilon_{1t+1} E_{t+1} \varepsilon_{1t+2} = 0,
 \end{aligned}$$

$$E_t R_{1t+1}^{-1} \varepsilon_{1t+2} = E_t E_{t+1} R_{1t+1}^{-1} \varepsilon_{1t+2} = E_t R_{1t+1}^{-1} E_{t+1} \varepsilon_{1t+2} = 0.$$

Substituting these 2 equalities into (5) gives

$$R_{2t}^{-1} = R_{1t}^{-1} \left[E_t R_{1t+1}^{-1} + \text{cov}_t \left(R_{1t+1}^{-1}, \varepsilon_{1t+1} \right) \right], \quad (6)$$

where
$$\varepsilon_{1t+1} = \beta \frac{(\partial U(c_{t+1}) / \partial c_{t+1})}{(\partial U(c_t) / \partial c_t)} R_{1t} - 1.$$

For example,

$$\uparrow R_{1t+1}^{-1} \Rightarrow \downarrow c_{t+2}, \uparrow c_{t+1} \Rightarrow \downarrow \frac{\partial U(c_{t+1})}{\partial c_{t+1}} \quad (\text{since utility is concave}) \Rightarrow \downarrow \varepsilon_{1t+1},$$

hence,
$$\text{cov}_t \left(R_{1t+1}^{-1}, \varepsilon_{1t+1} \right) < 0, \quad \text{so from (6),}$$

$$R_{1t} = R_{2t} \cdot \left[E_t R_{1t+1}^{-1} + \text{cov}_t \left(R_{1t+1}^{-1}, \varepsilon_{1t+1} \right) \right],$$

$$R_{1t} E_t R_{1t+1} = R_{2t} + \text{cov}_t \left(R_{1t+1}^{-1}, \varepsilon_{1t+1} \right) R_{2t} E_t R_{1t+1} < R_{2t}$$

Eq.(6) is a generalized version of the pure expectation theory of the term structure of interest rates, adjusted for the risk premium $\text{cov}_t(R_{1t+1}^{-1}, \varepsilon_{1t+1})$.

If the private agent is risk averse, then $\text{cov}_t(R_{1t+1}^{-1}, \varepsilon_{1t+1}) < 0$.

If the private agent is risk lover, then $\text{cov}_t(R_{1t+1}^{-1}, \varepsilon_{1t+1}) > 0$.

The pure expectation theory states that $R_{2t}^{-1} = R_{1t}^{-1} [E_t R_{1t+1}^{-1}]$, this is the result from the assumption that private agent is risk neutral.

Contingent Claim Market

Pricing of one-step-ahead state-contingent securities.

- **Let the state of the economy evolve according to a Markov process described by density $f(x',x)$, hence,**

$$prob(x_{t+1} \leq x' | x_t = x) = \int_{-\infty}^{x'} f(u, x) du \equiv F(x', x)$$

Contingent Claim Market

- In period t , given that the economy is in state x_t , then one can purchase, or sell, a claim to 1 unit of next period consumption good contingent on the event that x_{t+1} belongs to a set A , at the following price (measured in terms of time t consumption)

$$\int_{x_{t+1} \in A} q(x_{t+1}, x_t) dx_{t+1}$$

Contingent Claim Market

- Let Ω be the entire space of possible x , the time t , price of a perfectly certain claim on period $(t+1)$ consumption is

$$\int_{x_{t+1} \in \Omega} q(x_{t+1}, x_t) dx_{t+1} = \frac{1}{R_{1t}}$$

*= the price of a perfectly sure claim to
1 unit of consumption at time $t+1$,*

where

$$\int_{x_{t+1} \in \Omega} q(x_{t+1}, x_t) dx_{t+1} > \int_{x_{t+1} \in A} q(x_{t+1}, x_t) dx_{t+1}.$$