1.3 Concave functions.

- 1. In economics, we often see concave real-valued functions when domains are convex sets.
- This concavity concept is more general than the differential approach to find maximum. It holds even when the function is not differentiable. However, this is more difficult to verify.
- 2. f is a concave function iff for every pair of distinct points <u>on its graph</u>, the chord joining them lies on or below the graph. Here, we look at the value of the function evaluated at convex combinations of any two points, $f(\mathbf{x}^t)$, must be greater than or equal to convex combination of $f(\mathbf{x}^0)$ and $\mathbf{f(x^1)}$.

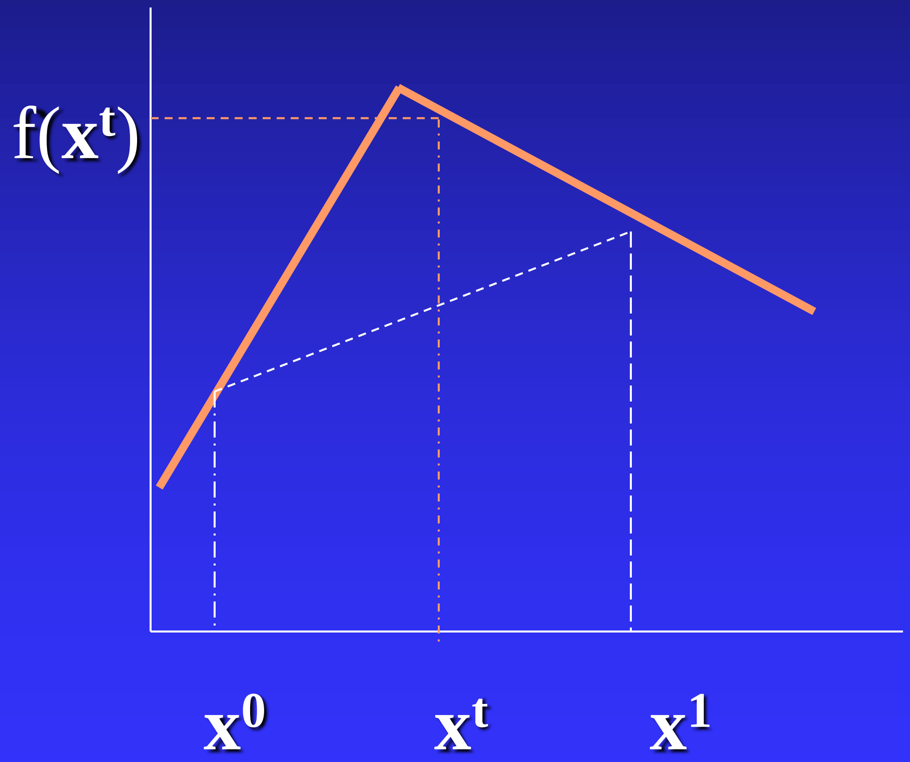
Concave Functions

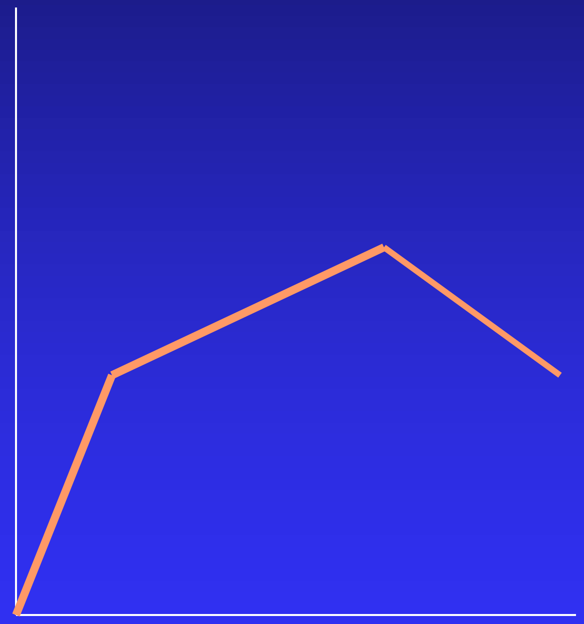
- $f(\mathbf{x}^t) \geq \; t\, f(\mathbf{x}^0) + (1\text{-}t)\, f(\mathbf{x}^1)$, for $0 < t < 1$.
  chord

A concave function

Concave Functions

This definition is quite general since it allows for kinks in f(x) which is not differentiable.





Concave functions.

- 3. Another equivalent definition <u>when f is also continuously differentiable</u>, we say f is concave iff for all $\mathbf{x} \in D$, $f(\mathbf{x}^2)-f(\mathbf{x}^1) \leq Df(\mathbf{x}^1)(\mathbf{x}^2-\mathbf{x}^1)$
- Equivalently

$$y = f(\mathbf{x})\, is\, concave\, iff \;\; for\, any\, two\, distinct\;\, points$$

$$\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2)\; and\; \tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2)$$

$$f(\bar{\mathbf{x}}) \leq f(\tilde{\mathbf{x}}) + \frac{\partial f(\tilde{\mathbf{x}})}{\partial x_1}(\bar{x}_1 - \tilde{x}_1) + \frac{\partial f(\tilde{\mathbf{x}})}{\partial x_2}(\bar{x}_2 - \tilde{x}_2).$$

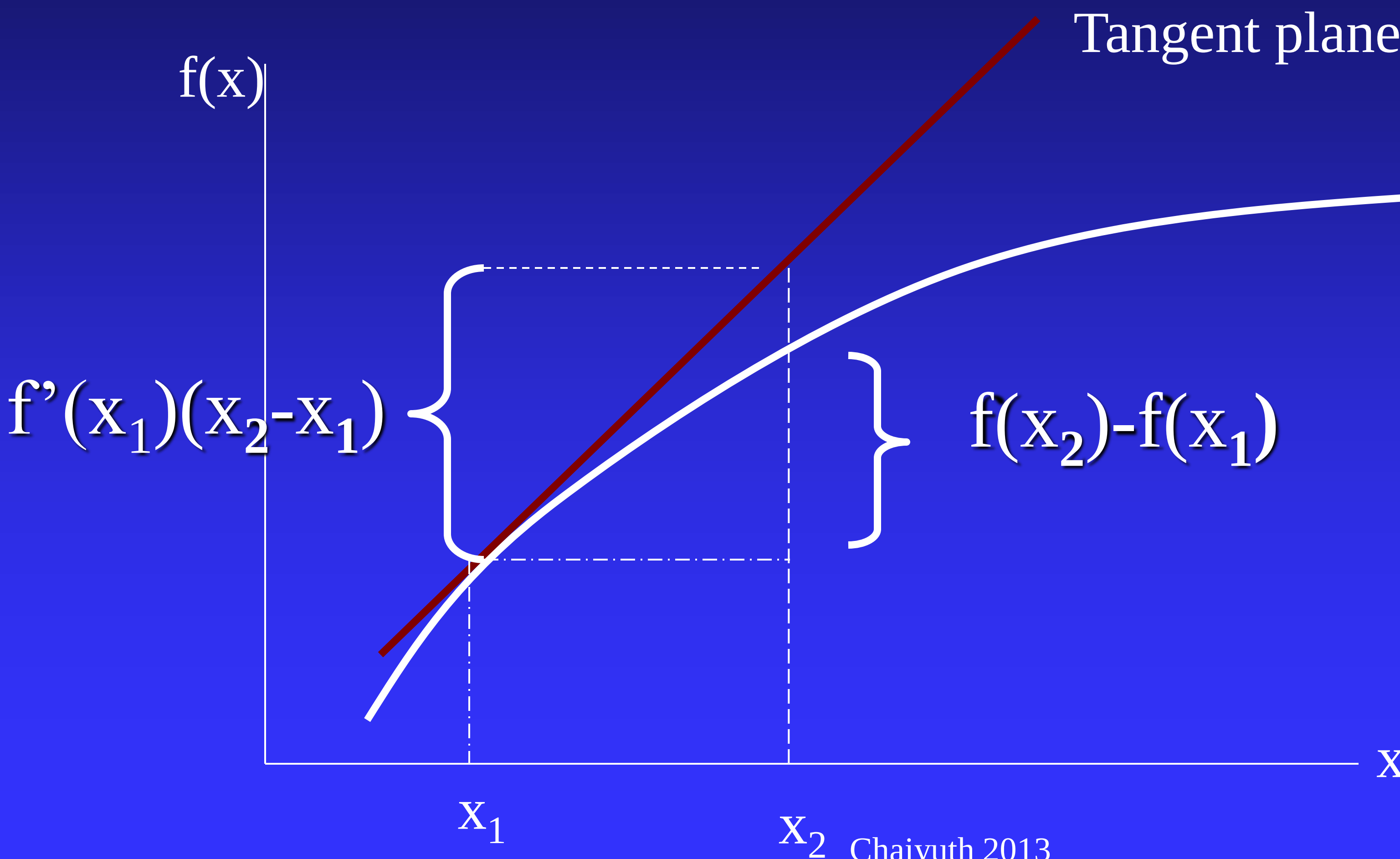
Concave functions.

- This means that a tangent plane to the graph of f(x) must remain everywhere above the graph.
- Or you could say the graph of f(x) must curve away from its tangent plane in every direction.
- The equation for the tangent plane at x_1 is

$$f(\mathbf{x_1}) + f\,'(\mathbf{x_1})(\mathbf{x_2}\text{-}\mathbf{x_1})$$

Concave functions.

- $f(\mathbf{x_2}) - f(\mathbf{x_1}) \leq f\,'(\mathbf{x_1})(\mathbf{x_2}\text{-}\mathbf{x_1})$.

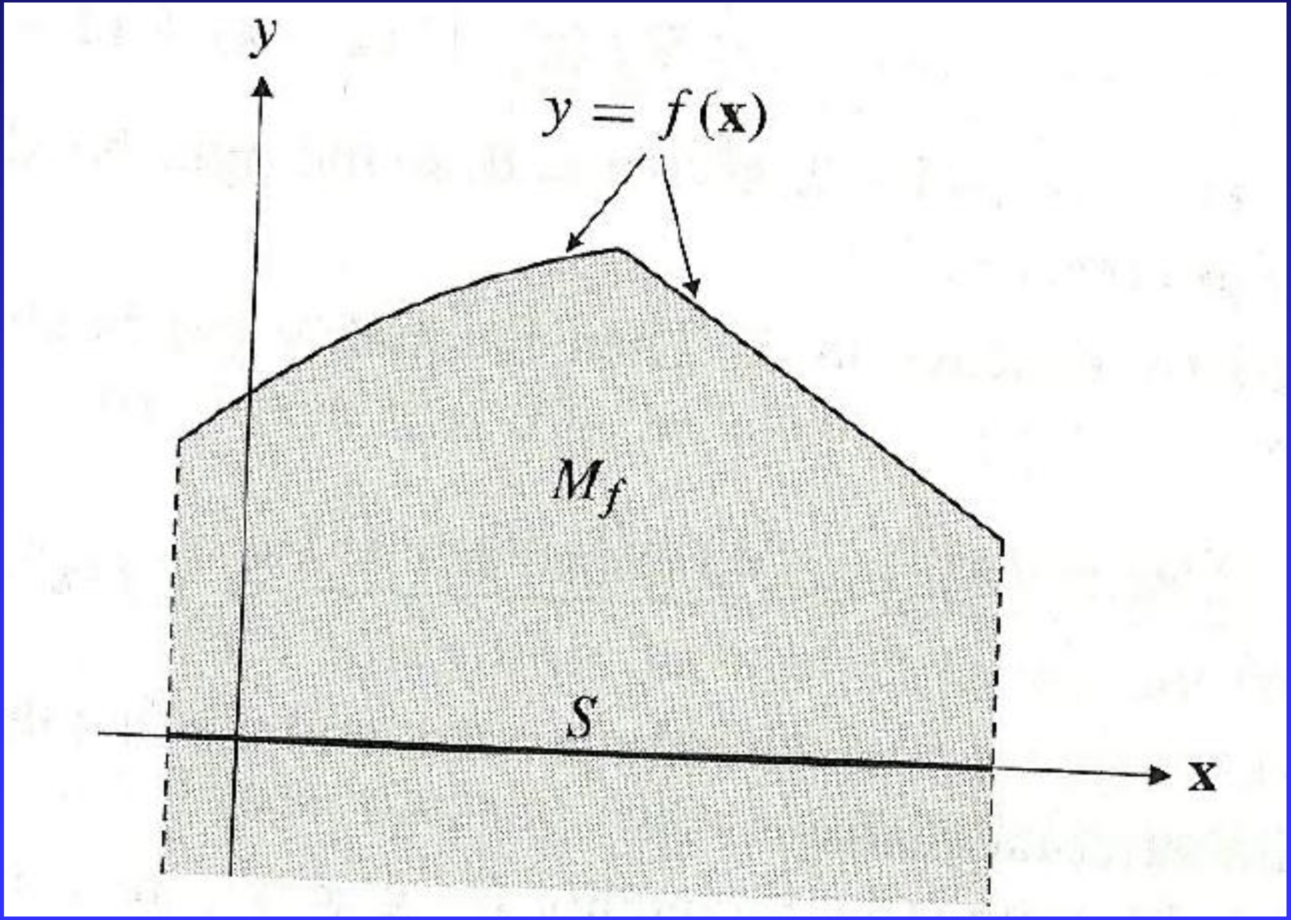


Concave functions.

- 3.1 When f is twice continuously differentiable, f is concave iff the Hessian matrix of a function is negative semidefinite at every point. We will discuss how to determine definiteness of matrix later.
- 3.2 Another useful definition: f is a concave function iff the set of points on and below (i.e. beneath) the graph is a convex set. Equivalently, we look at the shape of set $M_f = \{(x, y) / x \in S, f(\mathbf{x}) \geq y\}$.

Concave functions.

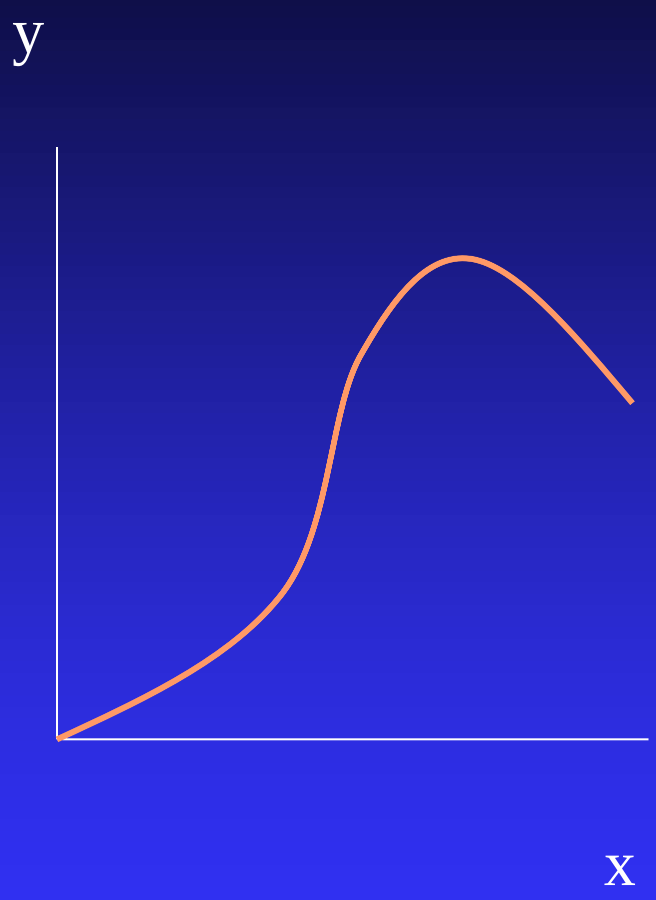
- $M_f = \{(x, y) / x \in S, f(\mathbf{x}) \geq y\}$.



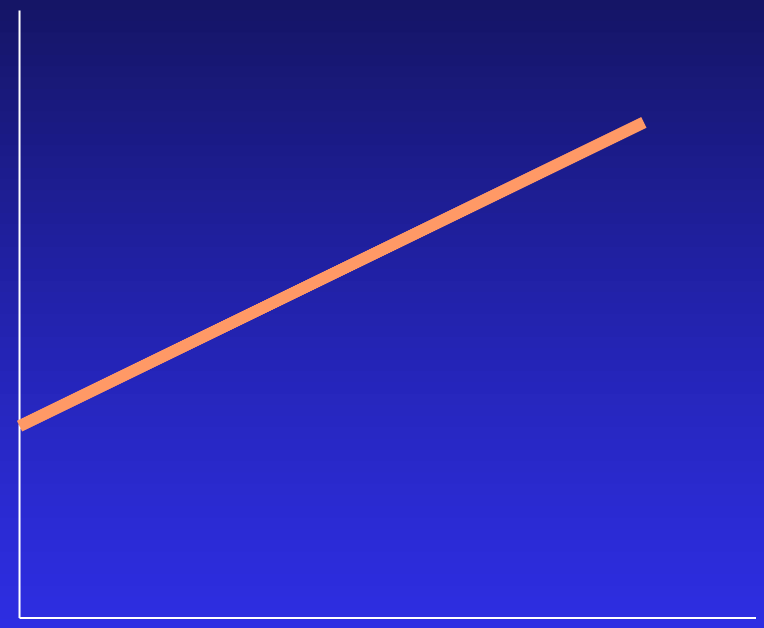
Concave functions.

- 4. A concave function allows for linear segments. To rule out this, we require a strictly concave function.
- 5. "Strictly" normally is equivalent to get rid of equality sign in definition. For $0 < t < 1$,

$$f(\mathbf{x}^t) > t \,.\, f(\mathbf{x}^0) + (1-t).\, f(\mathbf{x}^1)$$



Not concave



Not strictly concave

Concave functions.

- 6. So, f is a strictly concave function iff the chords joining any pairs of points must lie below the graph.
- 7. Nice properties of the concave functions are

  (1) the critical points are always global maxima; If strictly concave, FOC yields a unique global maximum.

  (2) sum of concave functions is concave;

Concave functions.

- (3) their level sets have just the right shapes; they bound convex subsets from below. Also, the upper contour set is a convex set.
- For (2), we need it for the grand utility or social welfare function.
- For (3), it has nice interpretations, diminishing marginal rate of substitution, and mixing goods make you happier.

Concave functions.

- Def. g is a monotonic transformation of I if g is strictly increasing function of I.
- (4). Generally, a monotonic transformation of a concave function needs not be concave.
- In fact, any monotonic transformation of a concave function is a quasiconcave function. Think of $f(x) = x$, and $g(x) = x^2$.
- But an <u>increasing concave</u> function of a concave function is concave. For example, if f is a concave function, for $k > 0$, then kf is a concave function.

Concave functions.

- Example. $U(x,y)= x\ y$.
- Then, $3x\ y+2$, $(x\ y)^2$, $(x\ y)^2+2$, $\ln(x\ y)$ are its monotone transformation.
- We use this property for an ordinal concept of utility.

1.4 Quasiconcave functions

- 1. f is a quasiconcave function iff the superior set is a convex set.
- 2. The superior set is the set containing points in domain that gives the function a value ≥ a specific value of y. So, it is the area on and above the level set of a given value of y.

Quasiconcave functions

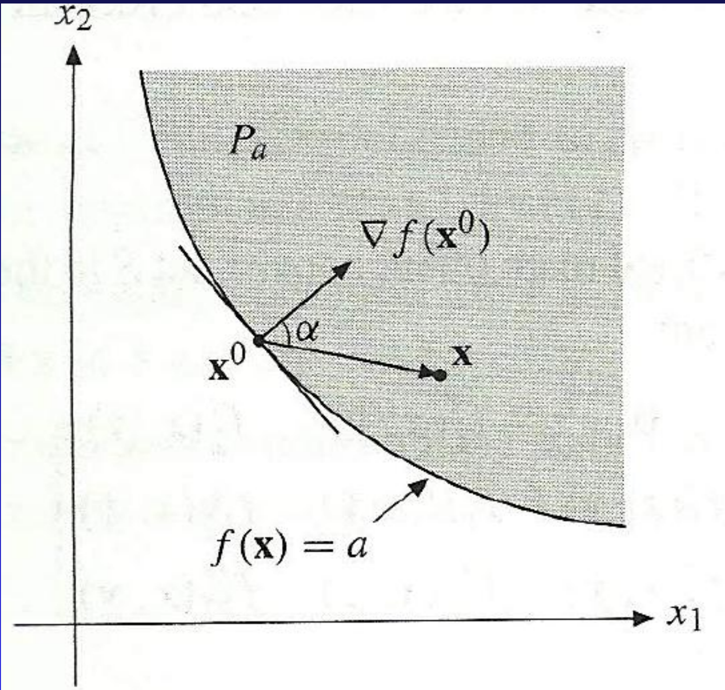


Figure 5 P_a is convex and $\nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \geq 0$.

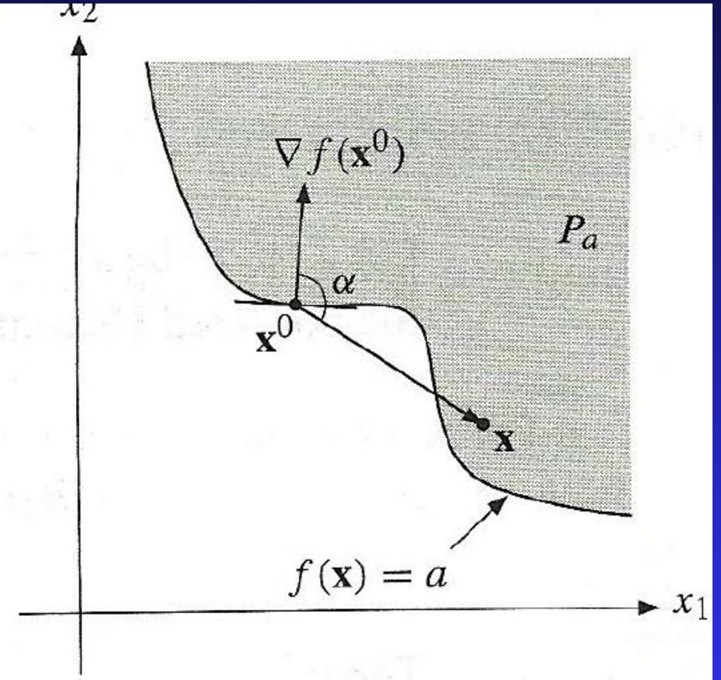
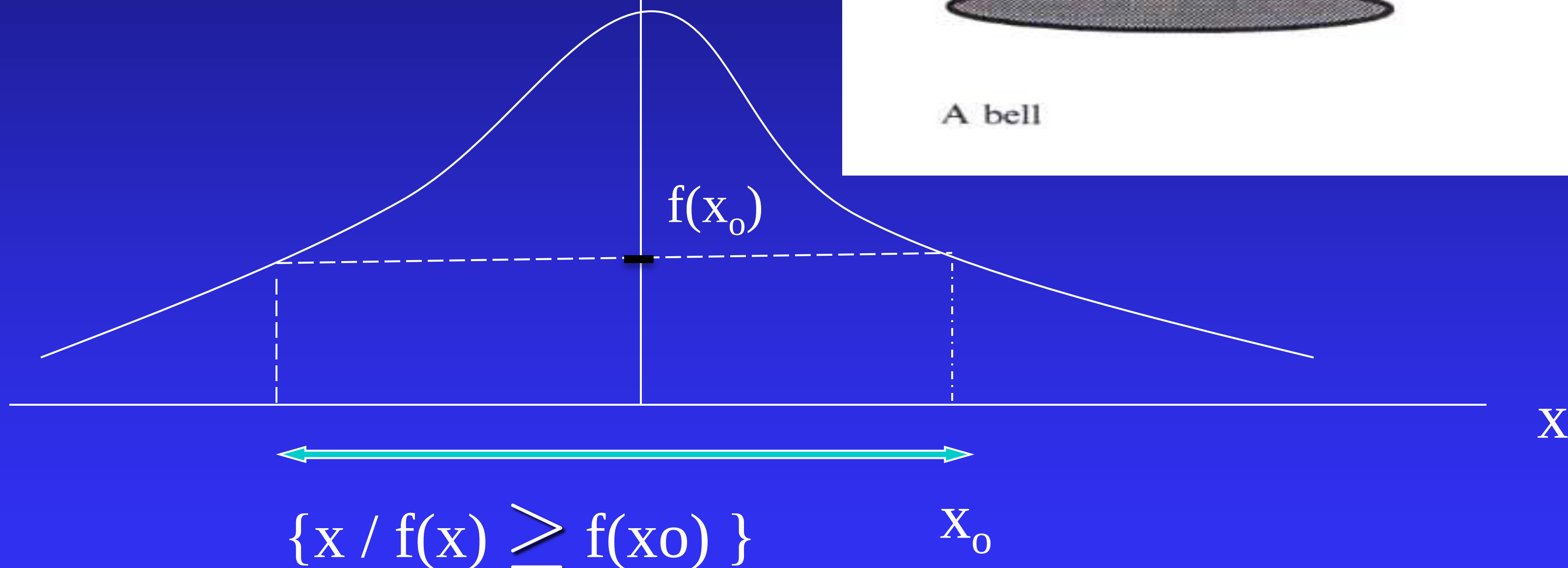
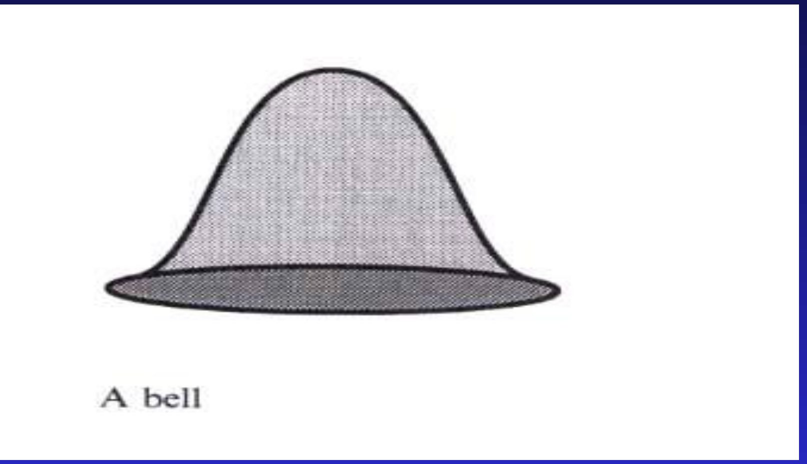


Figure 6 P_a is not convex and $\nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) < 0$.

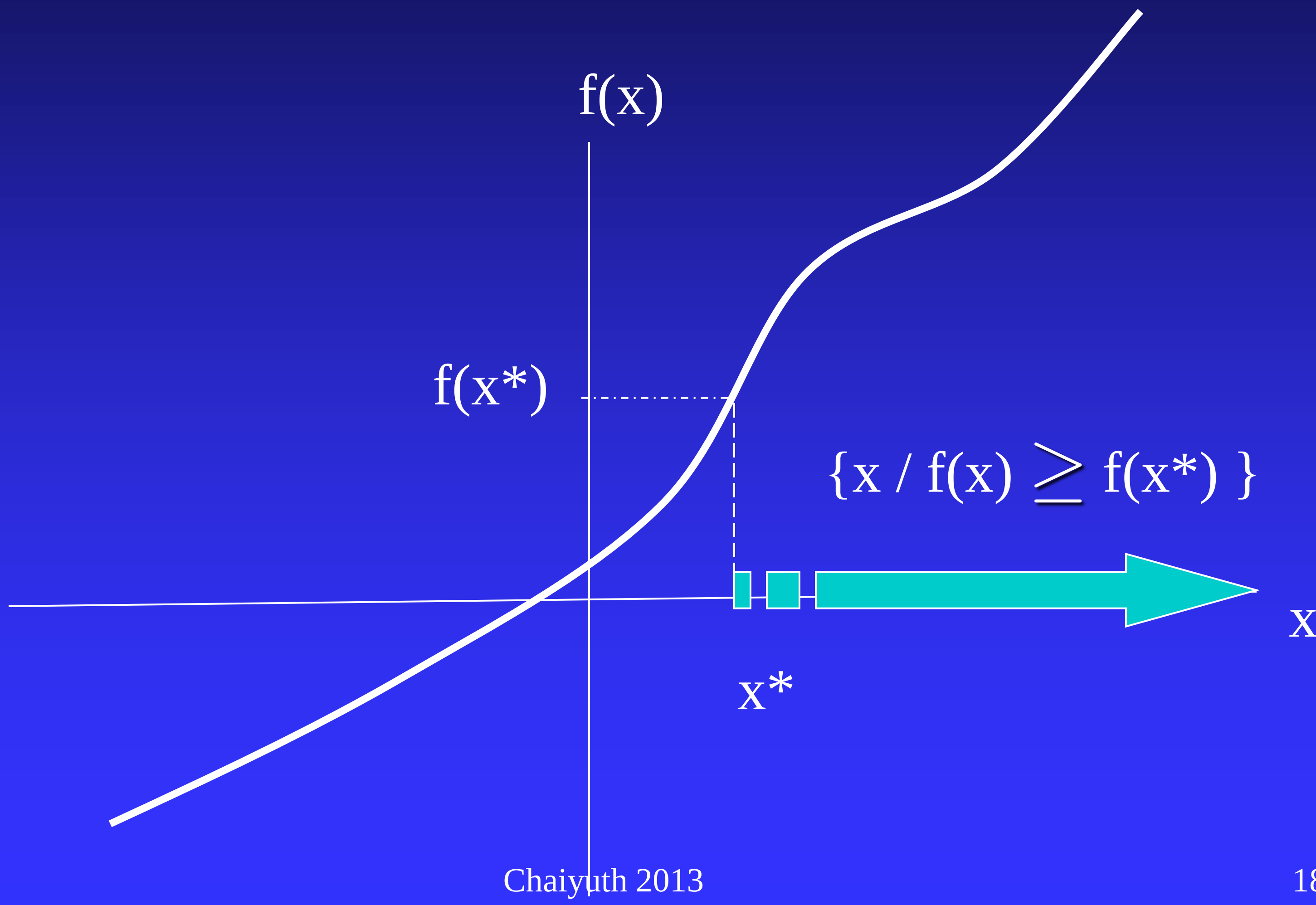
Quasiconcave functions





Quasiconcave functions

An increasing function on R^1 is quasiconcave.



Quasiconcave functions

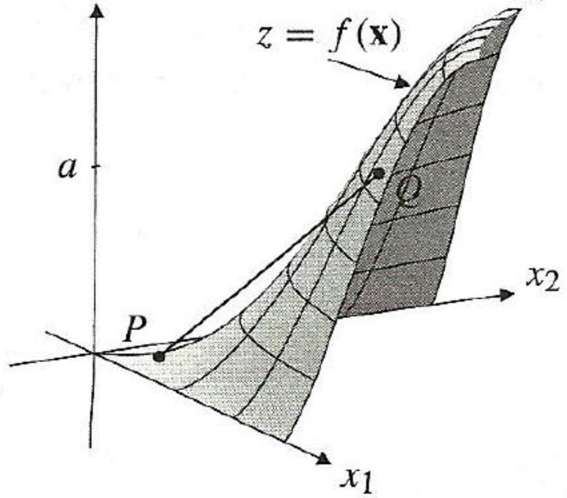


Figure 1 A quasiconcave function of two variables.

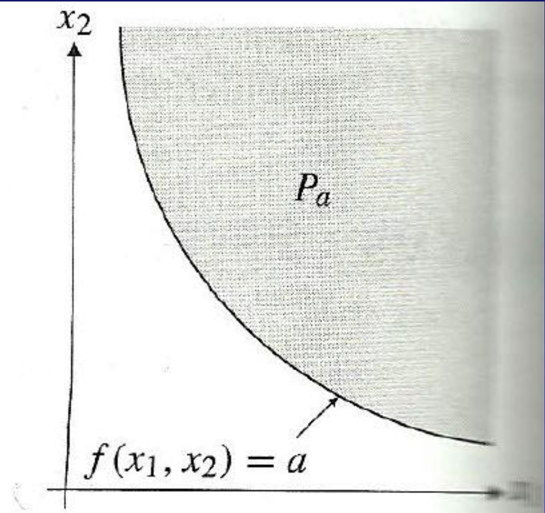
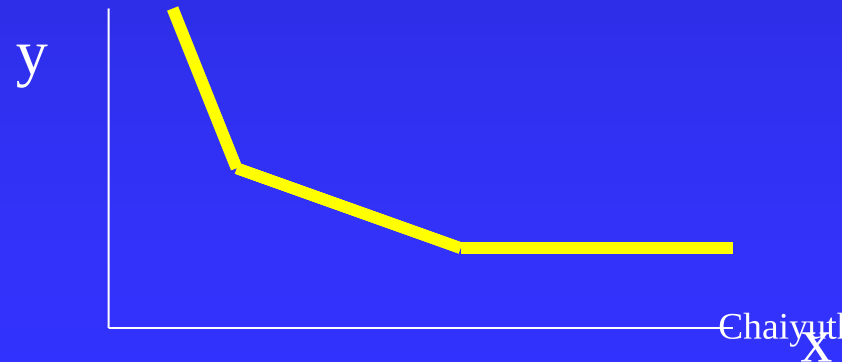


Figure 2 An upper level set for the function in Fig. 1.

Quasiconcave functions

- Ex. $F(x, y) = \min\{ ax, by\}$ with $a, b > 0$.

- 

Quasiconcave functions

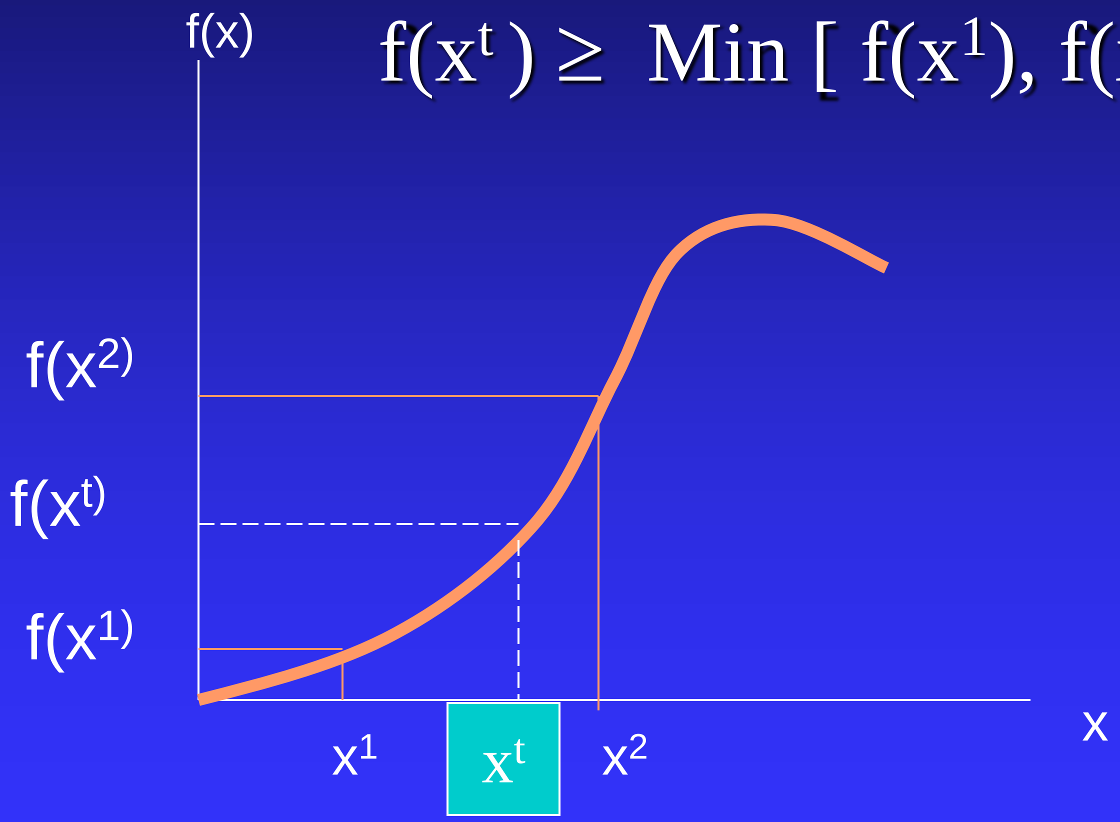
- 3. Another definition is to look at the value of the function: the value of the function at points formed by convex combination must be greater or equal the lowest value of the functions at any two points.
- If $f(\mathbf{x}^1) < f(\mathbf{x}^2)$, then $f(\mathbf{x}^t) \geq f(\mathbf{x}^1)$, for all $t \in [0, 1]$, and $\mathbf{x}^t \equiv t\,\mathbf{x}^1 + (1-t)\mathbf{x}^2$,

$$f(\mathbf{x}^t) \geq \text{Min}\,[\, f(\mathbf{x}^1), f(\mathbf{x}^2)\,], \text{ for all } t \in [0,1].$$

Read "the smaller of .."

Quasiconcave functions

$$f(x^t) \geq \text{Min}\,[\, f(x^1), f(x^2)\,]$$

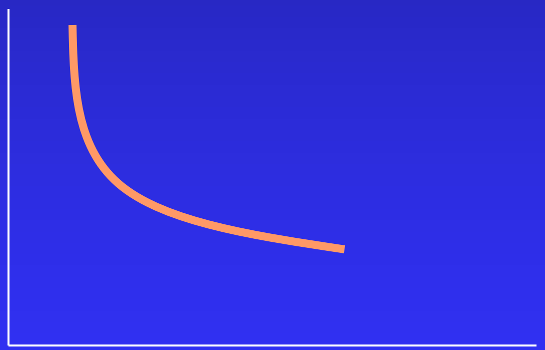


Quasiconcave functions

- 4. Noting that if the level set has some linear portion, it is still a convex set. So, if we want to get rid of this linear segment of the level curve, we will require a strictly quasiconcave function.

5. To remember this, if we assume that the utility function is strictly quasiconcave, then their indifference curves have no linear segments.

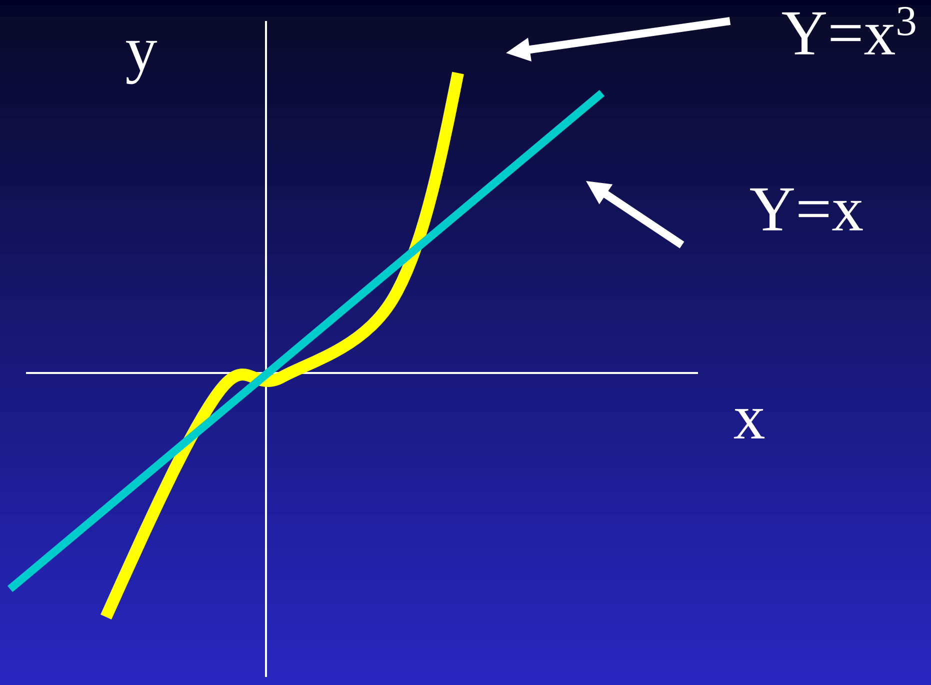
A quasi-concave function



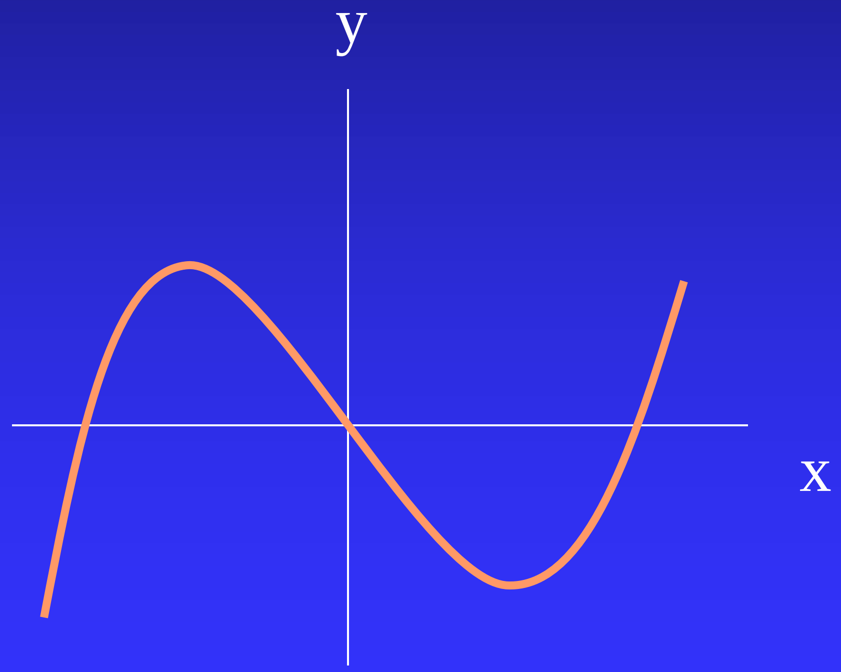
A strictly quasi-concave function

Quasiconcave functions

- 6. Some properties of quasiconcave functions:
- (1) sum of quasiconcave functions is not necessary quasiconcave;
- (2) a critical point need not be a maximum. For (1), think of $z=x^3-x$, we know that both x^3 and $-x$ are both monotone in R, so they are quasiconcave, but z, a sum of quasiconcave functions is neither quasiconcave nor quasiconvex.



$Y=x^3-x$

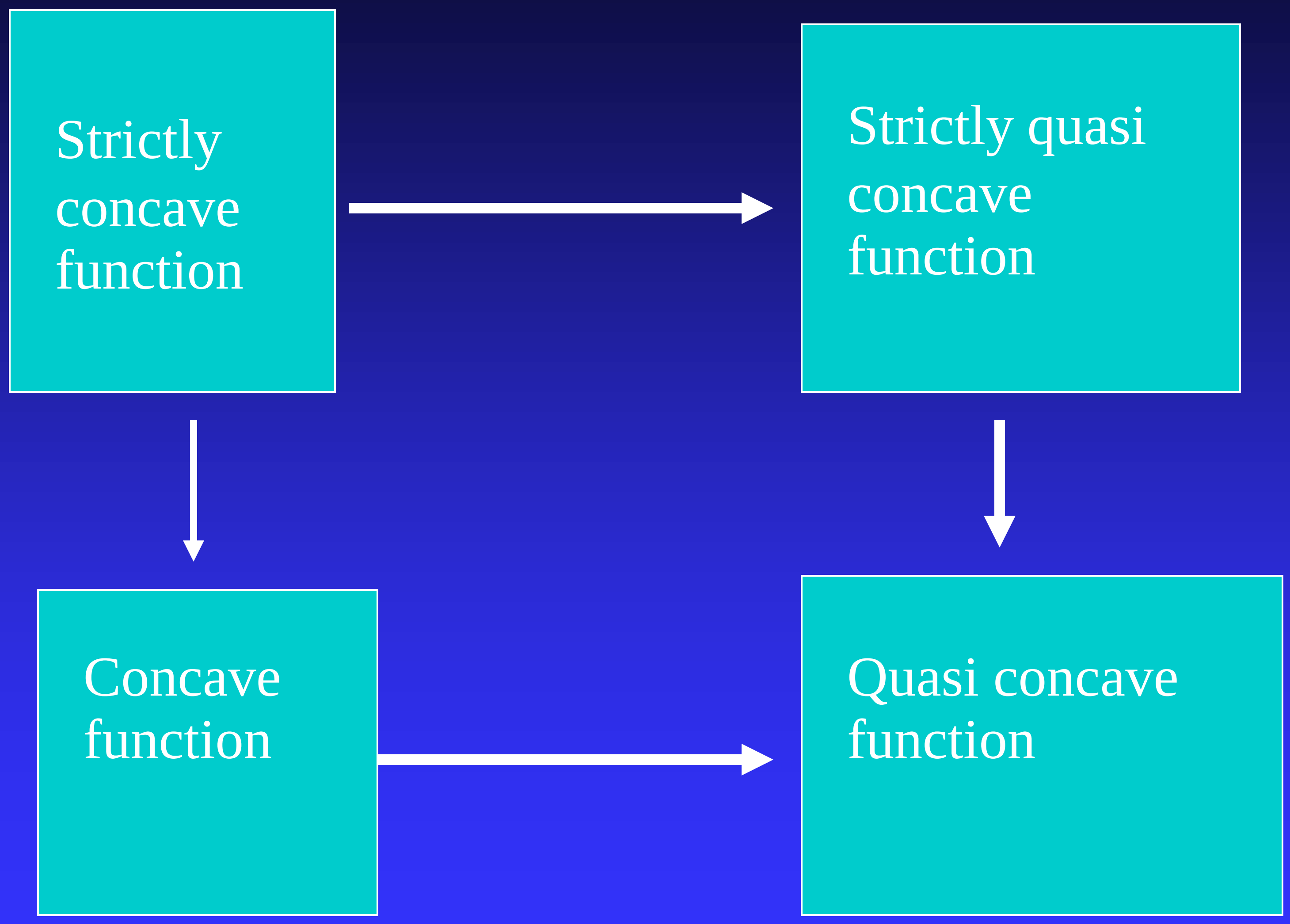


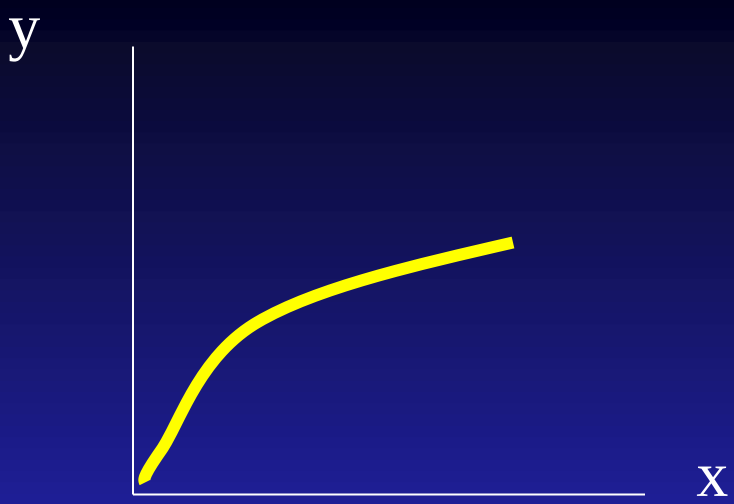
Concavity and quasiconcavity

- 1. A concave function is always quasiconcave. So, does the strictly function.
- 2. The reverse is not true. I.e. the bell curve, graph of x^3 , a step function
- 3. From (1), we know that superior sets of a quasiconcave function are convex.
- 4. The linear graph is both concave and convex, thus quasiconcave and quasiconvex.

Concavity and quasiconcavity

- 5. So far, we should feel that we only need a function to be only a quasiconcave, if we need only a nice level curve that is convex to the origin, and a strictly quasiconcave for a nicer level curve.
- 6. A function f(x) is quasiconvex if –f(x) is quasiconcave.
- 7. If f(x) is quasiconcave and F(f) is an increasing function, then F(f(x)) is also quasiconcave.


Strictly concave function
Strictly quasi concave function
Concave function
Quasi concave function



Increasing and concave

Ex. $Y = Ak^a$ where $A>0$ and $0<a<1$

Decreasing, and concave

Checking Concavity

- Defining matrix definiteness.
- Some notation on partial derivative column vector

  $\nabla f(\mathbf{x}) \equiv (f_1(\mathbf{x}), \ldots, f_2(\mathbf{x}))$ called "gradient of f at $\mathbf{x}$". The gradient itself is a function.
- a matrix of second-order partials, Hessian matrix,

$$H(\mathbf{x}) \equiv \begin{pmatrix} f_{11}(\mathbf{x}) & f_{12}(\mathbf{x}) \\ f_{21}(\mathbf{x}) & f_{22}(\mathbf{x}) \end{pmatrix}$$

- When H is evaluated at certain points, H is just a matrix filled with numbers.

Checking Concavity

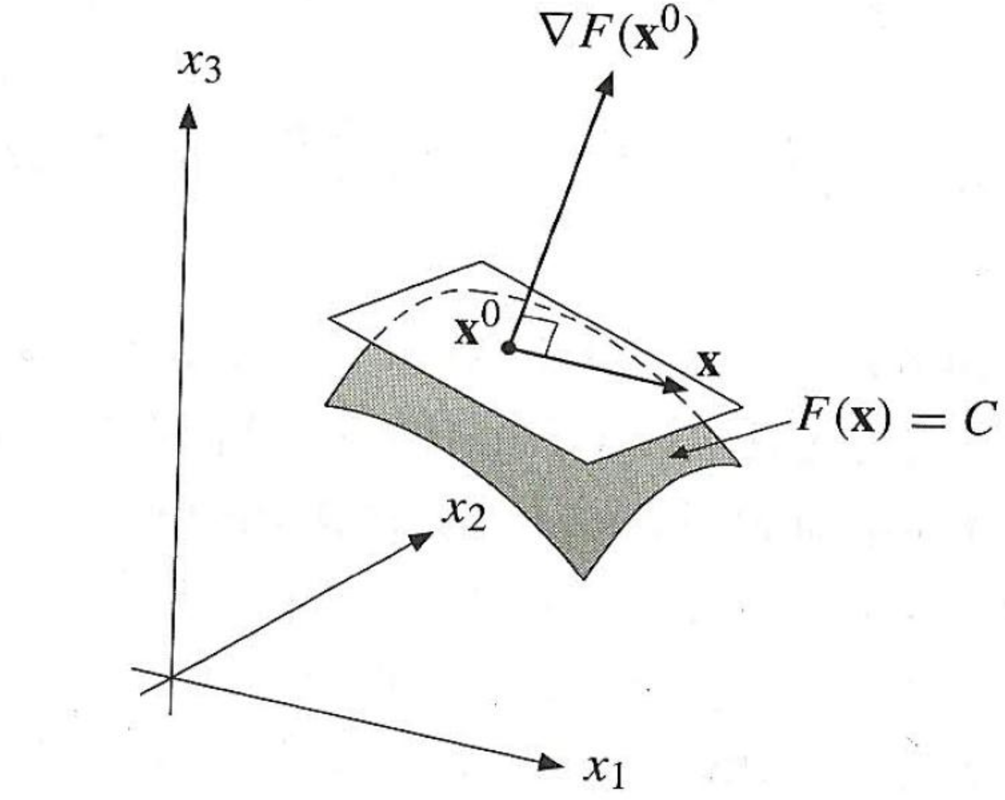


Figure 2 The gradient $\nabla F(\mathbf{x}^0)$ is orthogonal to the tangent plane of $F(\mathbf{x}) = C$ at $\mathbf{x}^0$.

Checking Concavity

- An (n x n) matrix **A** is called <u>negative semidefinite</u> if for all vector $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$,

$$\mathbf{x}'\mathbf{A}\mathbf{x} \leq 0.$$

- For 2x2, we have

$$(x_1 \;\; x_2) \begin{pmatrix} f_{11}(\mathbf{x}) & f_{12}(\mathbf{x}) \\ f_{21}(\mathbf{x}) & f_{22}(\mathbf{x}) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \leq 0.$$

- Think of how to know properties of a quadratic form

Checking Concavity: properties of a quadratic form

$$y = f(x_1, x_2)$$

$$dy = f_1 dx_1 + f_2 dx_2 \;\; and\; FONC : f_1(x_1, x_2) = 0\, and\;\; f_2(x_1, x_2) = 0$$

$$SOC : need\ to\ place\ sign\ restriction\ on\ d^2 y < 0$$

$$d^2 y = (f_{11} dx_1 + f_{21} dx_2) dx_1 + (f_{12} dx_1 + f_{22} dx_2) dx_2$$

$$= f_{11} dx_1^2 + 2 f_{12} dx_1 dx_2 + f_{22} dx_2^2$$

$$= (dx_1 \;\; dx_2) \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix}$$

$To\ link\ H\ and\ d^2 y, rewrite\ d^2 y$ into completing square term as

$$d^2 y = \underbrace{f_{11} dx_1^2 + 2 f_{12} dx_1 dx_2 + \frac{(f_{12})^2 dx_2^2}{f_{11}}} + f_{22} dx_2^2 - \frac{(f_{12})^2 dx_2^2}{f_{11}}$$

$$= f_{11} \left(dx_1 + \frac{f_{12} dx_2}{f_{11}} \right)^2 + \left(\frac{f_{11} f_{22} - (f_{12})^2}{f_{11}} \right) dx_2^2$$

$$d^2 y < 0\ if \;\; f_{11} < 0 \quad and \quad f_{11} f_{22} - (f_{12})^2 > 0.$$

Checking Concavity

- For example, we have

$$(x_1 \;\; x_2)\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \le 0.$$

- Get function $= -x_1^2 - x_2^2$, so for any values of $x \neq 0$, value of function is less than 0. So, A is negative semidefinite.
- In fact, A is also negative definite.
- If inequality is strict, then A is called <u>negative definite</u>.
- Example for negative semidefinite: $f = -(x_1+x_2)^2$

Checking Concavity

1. R^1 : f is concave iff f '' ≤ 0, for all x.
2. R^2 : f is concave iff the Hessian matrix is negative semidefinite for all x in domain.

- The simplest way to check this is to verify that for all x, $f_{11} \leq 0$ and $f_{22} \leq 0$.
- Note that f_{11} is just a change in slope of the function in the direction of x_1, keeping x_2 constant. And for f_{12} is just a curvature of the function when x_1 and x_2 change at the same time. This rule of thumb does not allow for both zero values of f_{11} and f_{22}.

Checking Concavity

2.1 The precise way to check for a concave function for two variables is to verify that (a) $f_{11} \leq 0$; and (b) $f_{11}\, f_{22} - f_{12}\, f_{21} \geq 0$.

2.2 Conditions (a) and (b) are equivalently stated as the leading principal minors of the Hessian matrix always alternate in sign, starting with negative.

2.3 Leading Principal minors are just the determinants of sub-matrices evaluated at the point **x**, as we move down the principal diagonal of the Hessian.

Checking Concavity

- For H is 2*2, $|H_1| = |f_{11}|$, $|H_2| = |H| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$

Ex. $f(\mathbf{x}) = x_2 - 4x_1^2 + 3x_1 x_2 - x_2^2$

$f_1 = -8x_1 + 3x_2$ and $f_2 = 1 + 3x_1 - 2x_2$

Solve for a critical points $x_1 = 3/7$; $x_2 = 8/7$.

$H = \begin{pmatrix} -8 & 3 \\ 3 & -2 \end{pmatrix}$, check leading principal minors

$|f_{11}| = -8$, $|H| = 16 - 9 = 7 > 0$

Checking Concavity

- 3. Hessian matrix tells us about the curvature of the function evaluated at certain points.
- 4. For a <u>strictly</u> concave function, we require the Hessian matrix to be negative <u>definite</u> for all x in domain set. That is, we need $f_{11} < 0$ and $f_{22} < 0$.
- Noting that we cannot claim that f is strictly concave iff H is negative definite. Since some strictly concave function have a H to be negative semi-definite at some points. For instance, $f(x_1, x_2) = -(x_1)^4 - (x_2)^2$. Check when both are zero. The right way to say is if H is neg. def, then f is strictly concave.

Checking Concavity

- 5. Example: $f(x_1, x_2) = x_1^{0.5}x_2^{0.5}$. We can verify that $f_{11} \leq 0$ and $f_{22} \leq 0$ for all x in domain. Thus, f is a concave function, and is also quasiconcave.
- 6. How about $f(x_1, x_2) = x_1x_2$. Here we have $f_{11} = 0$ and $f_{22} = 0$ and the determinant of the Hessian matrix (second order principal minor) is 0. In fact, this function is not neither concave nor convex.
- 7. Try $f = x_1^{\ 2} - x_2^2$

Checking Concavity

- 8. If you feel like to check negative definiteness of the Hessian that is larger than 2x2. We need the following condition:

If the leading principal minors of the Hessian matrix alternate in sign, starting with negative, then H is negative definite.

1.5 Homogenous functions

- 1. f is homogenous of degree k if $f(t\mathbf{x}) = t^k f(\mathbf{x})$, for all $t > 0$.
- 2. When k =1, f is also called linear homogenous.
- 3. $f(x) = Ax_1^a x_2^b$ is homogenous of degree a+b. This function is also known as Cobb-Douglas function.
- 4. If f is homogenous of degree k, its partial derivatives are homogenous of degree k-1. For example, consumer demand function is HD 0 in prices and income together; factor demand is HD 0 in all prices; marginal products depends on factor ratios, not absolute levels of factor use.

Homogenous functions: Euler's theorem

$f(\mathbf{x})$ is homogenous of degree k iff

$$kf(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial f(x)}{\partial x_i} x_i, \, for\, all\, x.$$

Proof.

$$f(x_1, x_2)\, is\, HD\, k \Leftrightarrow x_1 f_1'(x_1, x_2) + x_2 f_2'(x_1, x_2)$$

From definition, $f(tx_1, tx_2) = t^k f(x_1, x_2), \forall t > 0.$

Differentiate w.r.t. t gives

$$x_1 f_1'(tx_1, tx_2) + x_2 f_2'(tx_1, tx_2) = kt^{k-1} f(x_1, x_2).$$

for $t = 1$, $\quad x_1 f_1'(x_1, x_2) + x_2 f_2'(x_1, x_2) = kf(x_1, x_2).$

We can use this to show income distribution under the perfectly competitive model.

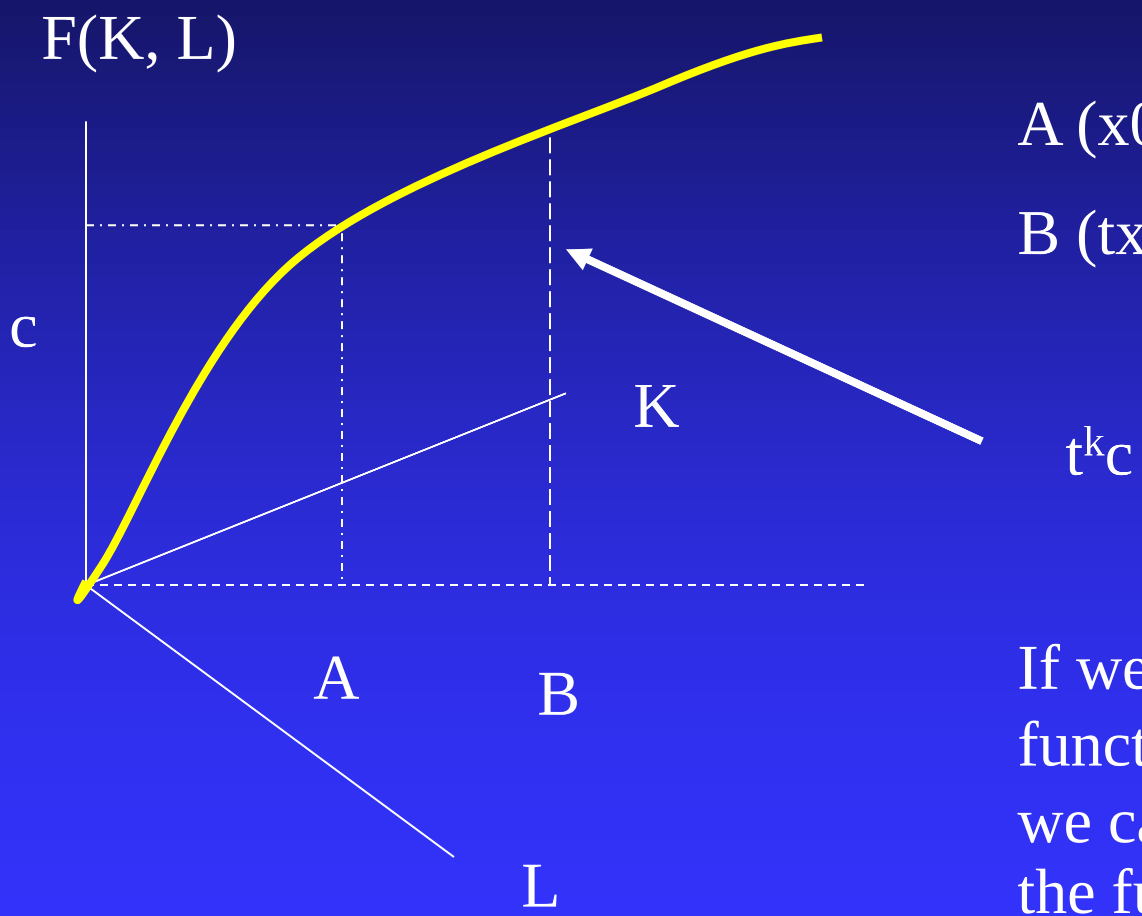
Homogenous functions

- 6. When k=1, we can write the linear homogenous function in terms of its partial derivatives.
- 7. The most useful one is when f is the production function using K and L:

$$f(K, L) = AK^{a} L^{1-a} = MPK * K + MPL * L.$$

- 8. Also, we can express the output-labor ratio as a function of the capital-labor ratio.

Homogenous functions



A (x0, y0)

B (tx0, ty0)

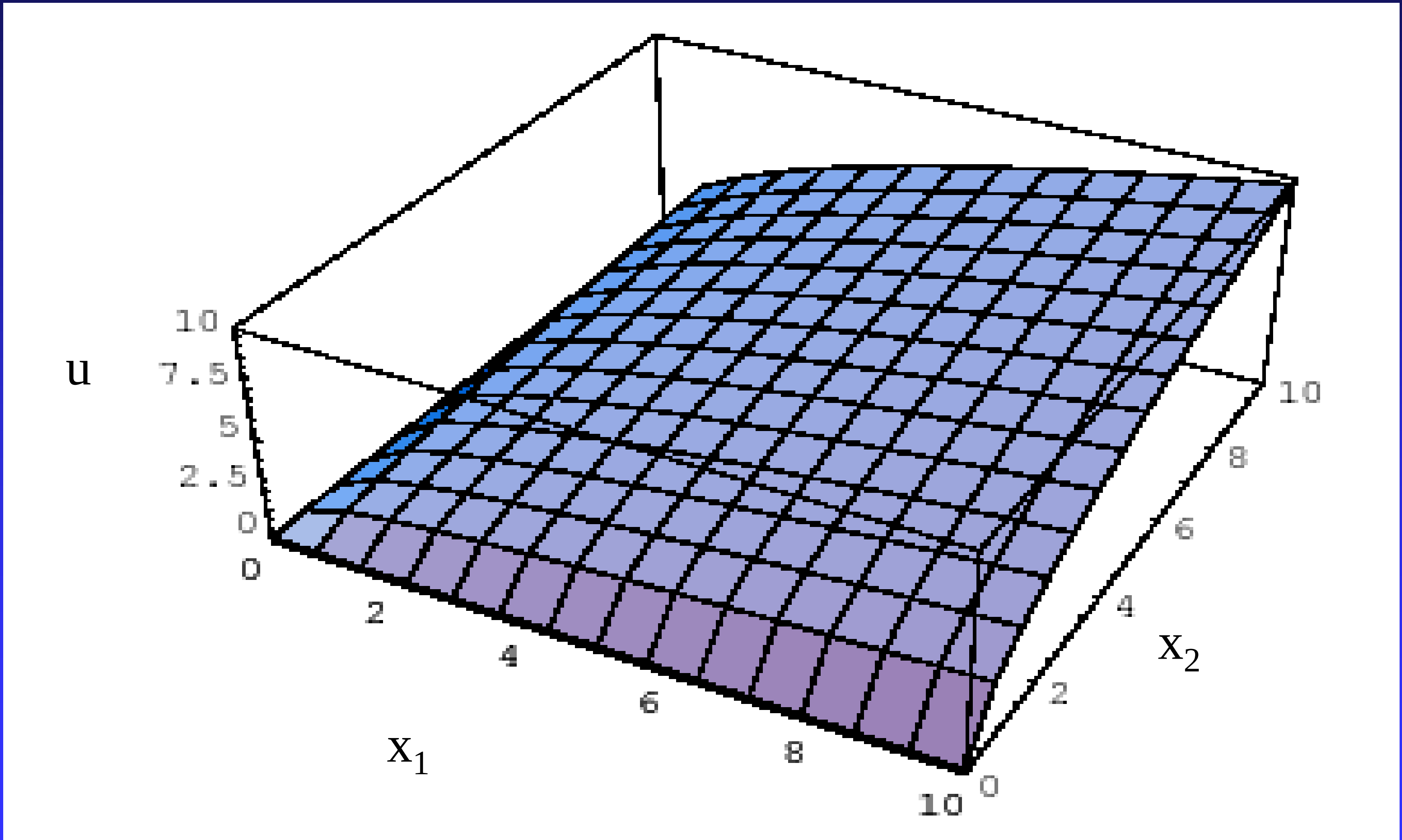
If we know the value of the function at one point, say A, we can determine all values of the function.

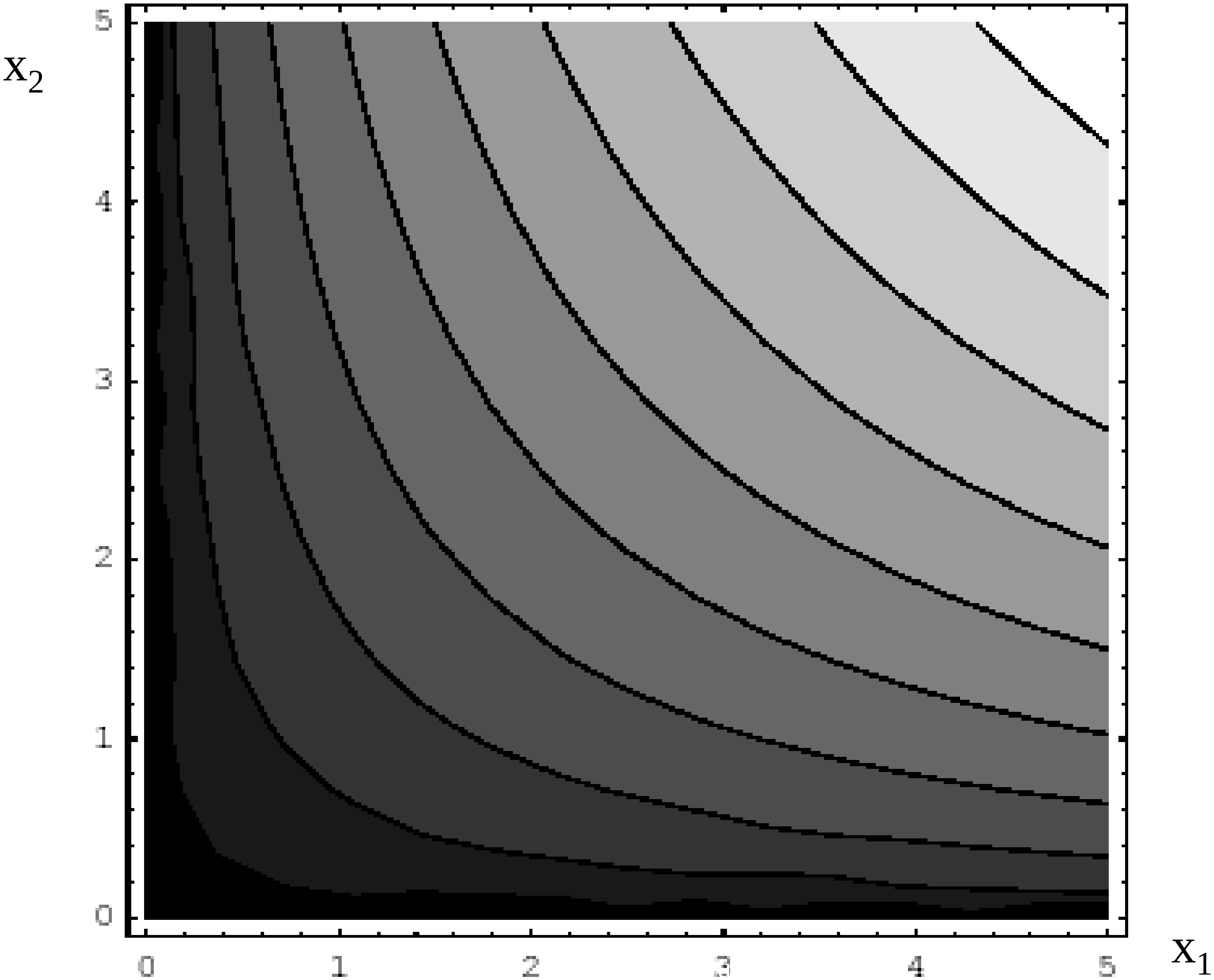
Useful theorems

- 1. If f is quasiconcave and linearly homogenous, then f is concave.
- 2. Every Cobb-Douglas function of two variables is quasiconcave.
- 3. CES function is quasiconcave since it is a monotonic transformation of a concave function.
- 4. A Cobb-Douglas function is concave iff it is CRTS or DRTS.

Nice examples of functions used in economics

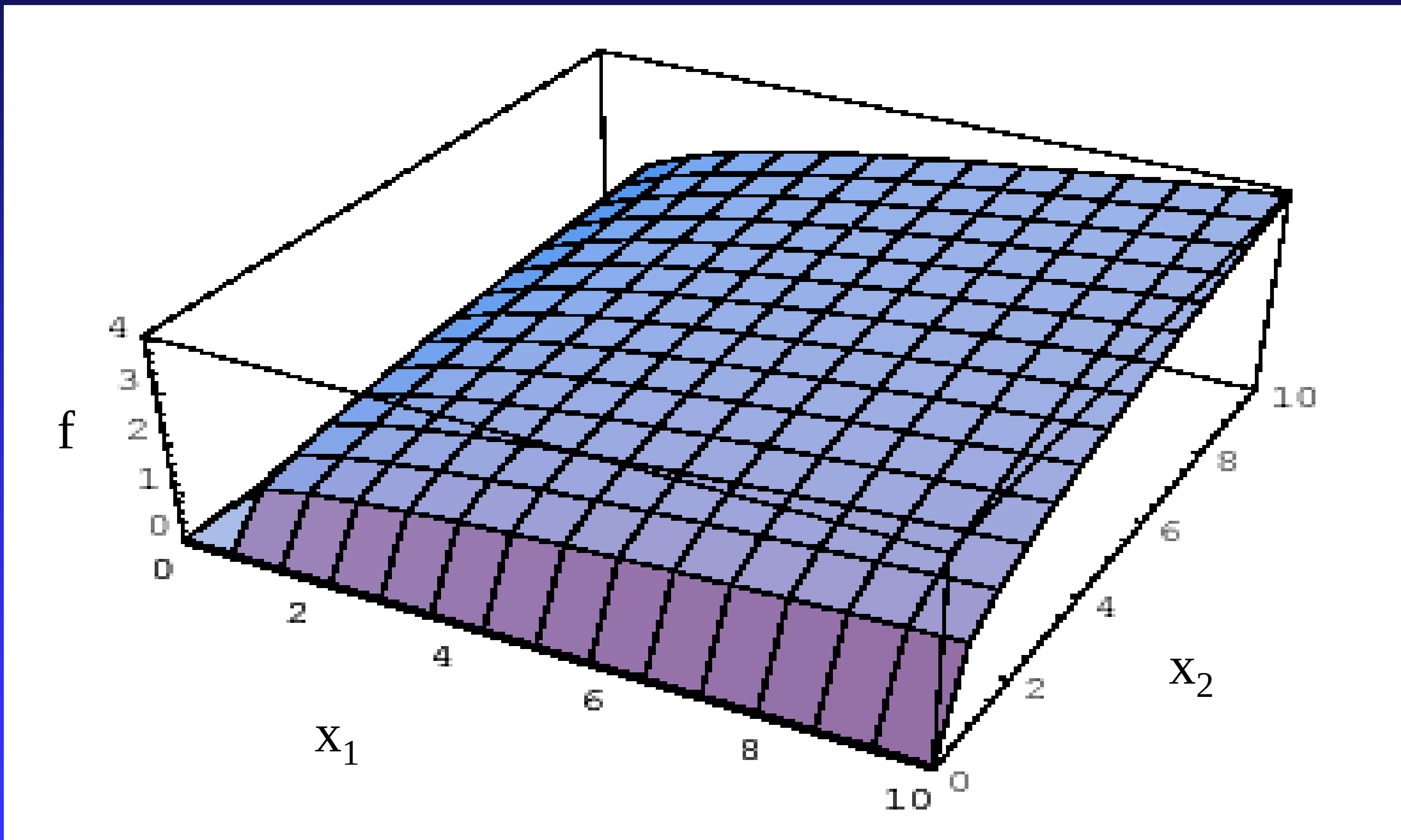
1. $u(x_1, x_2) = x_1^{(0.5)}\, x_2^{(0.5)}$



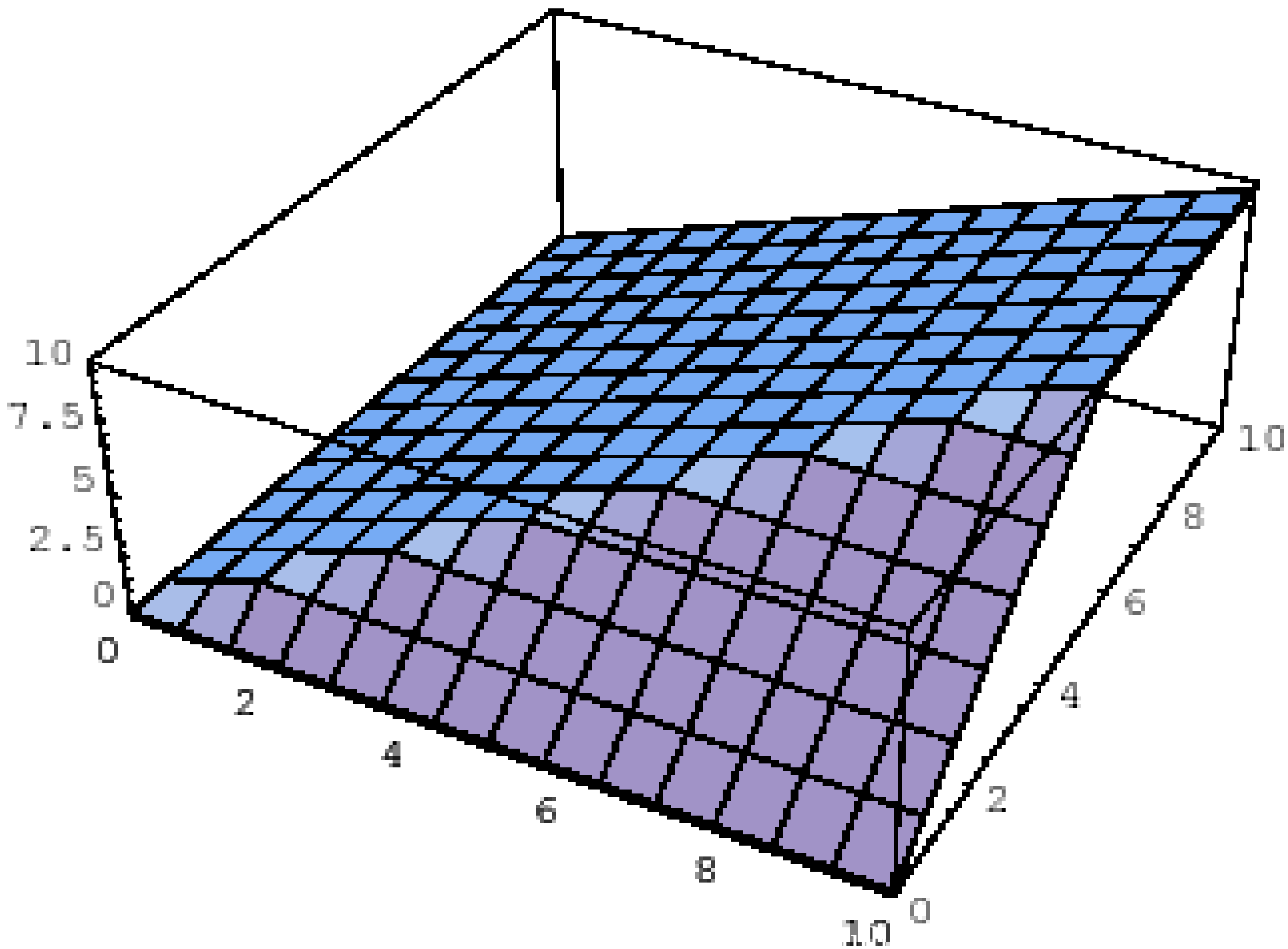


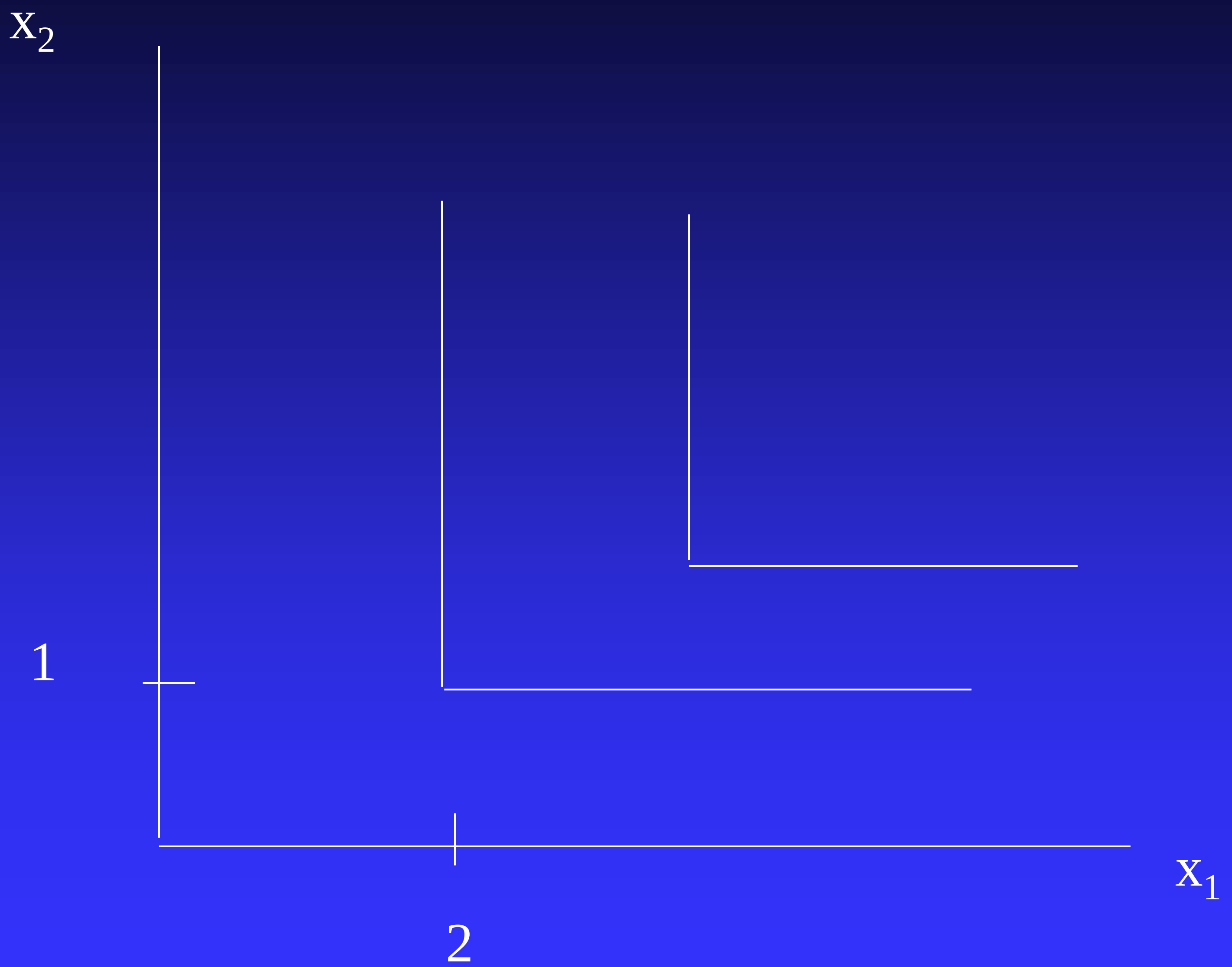
x_2
5
4
3
2
1
0
0
1
2
3
4
5
x_1

2. $f(x_1, x_2) = x_1^{(0.3)} x_2^{(0.3)}$



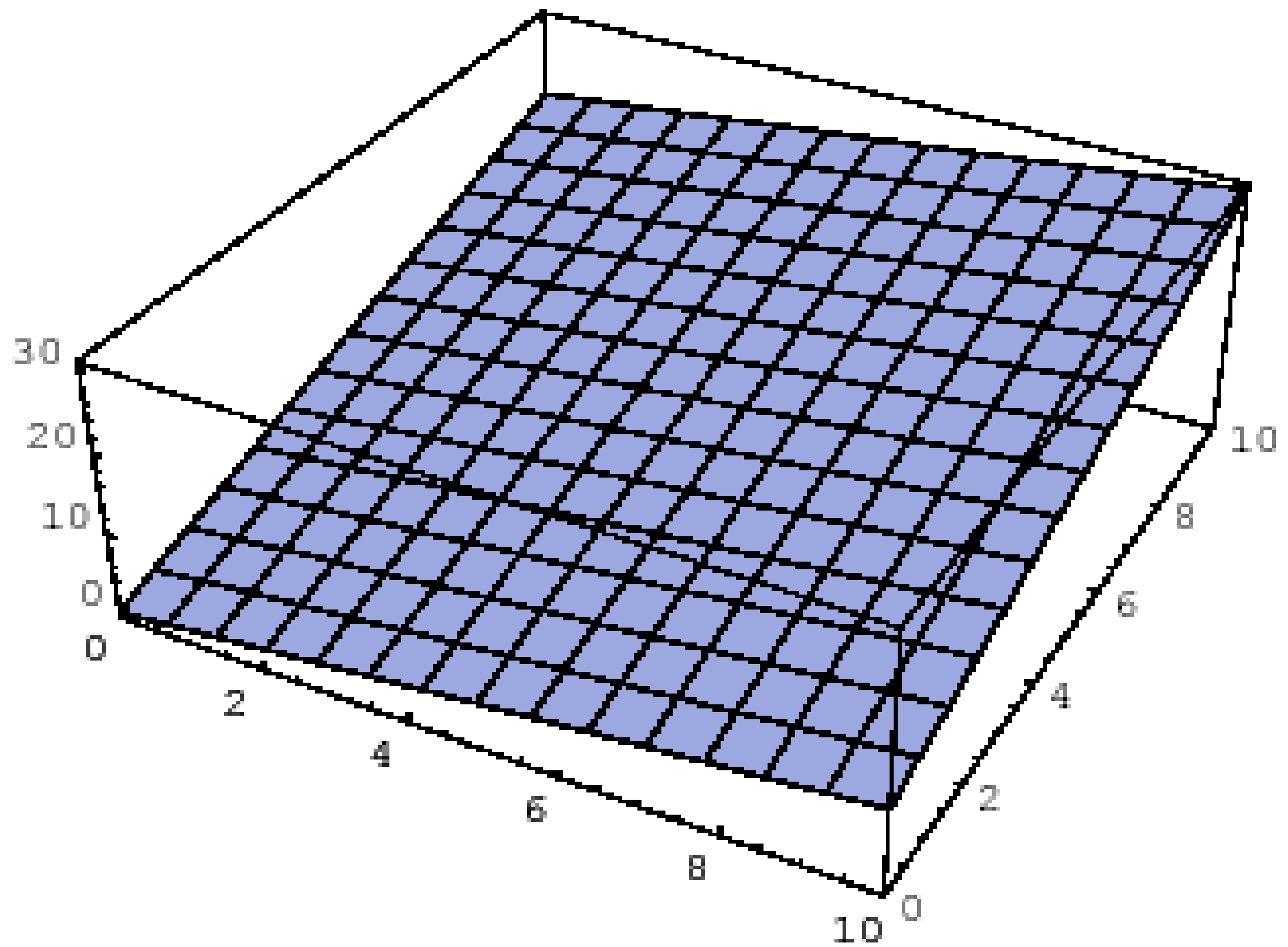
- 3. $f(x_1, x_2) = \min\{x_1, 2x_2\}$

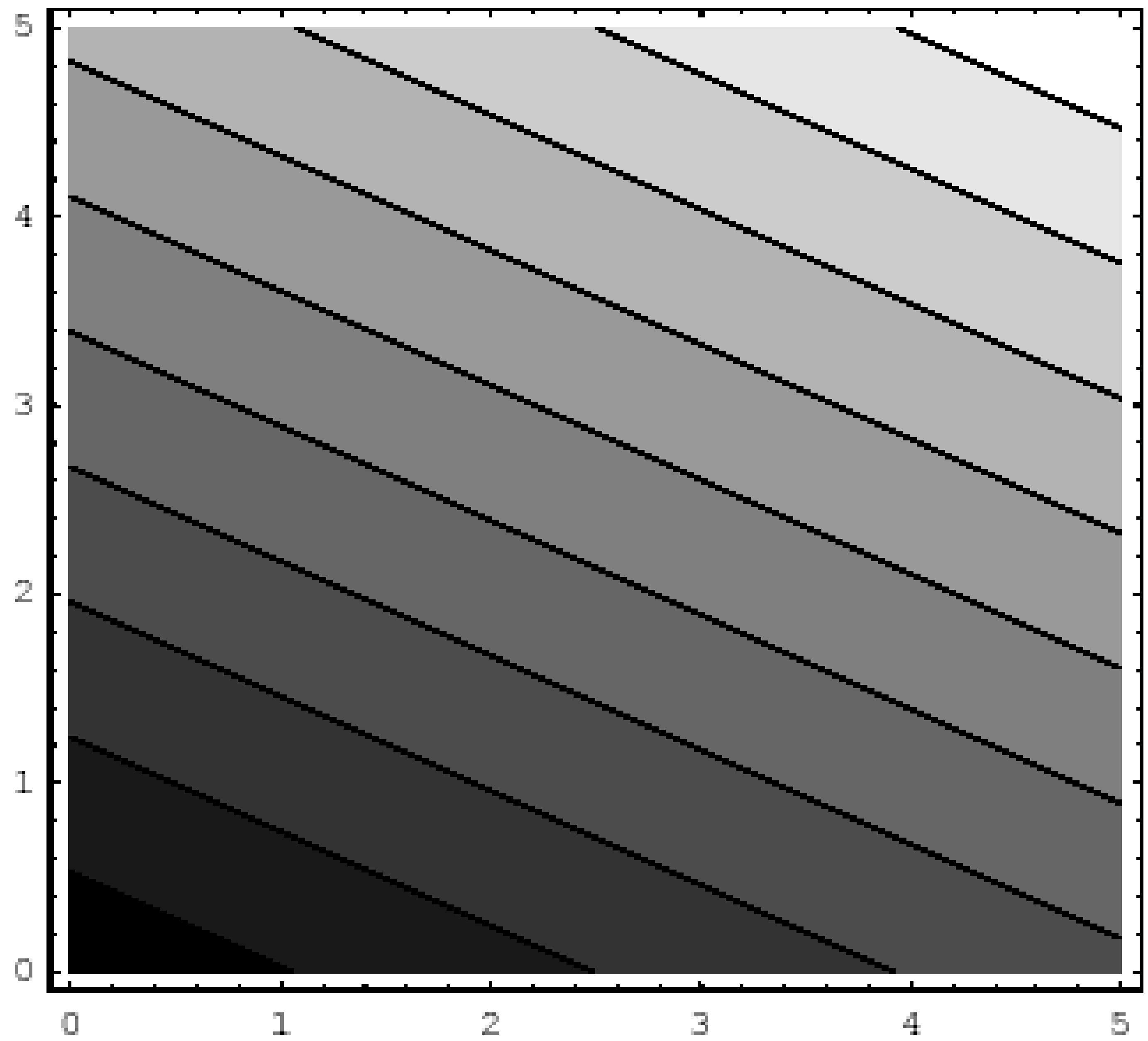




x_2
1
2
x_1

- 4. $f(x_1, x_2) = x_1 + 2x_2$





0
1
2
3
4
5
0
1
2
3
4
5

Some logical terms

- $p \rightarrow q$ we read this as follows:
  - if p, then q.
  - p is sufficient for q.
  - q is necessary for p.
  - p only if q.
  - q if p.

Some logical terms

- p $\leftarrow\rightarrow$ q we read this as follows:
- p if and only if q (= p if q , p only if q)
- p is equivalent to q.
- p is necessary and sufficient for p.