EE422 (1/2013)
Lecture 5
Introduction to Integration

Read - SYD ch.4

- Simon and Blume A.3

Skip multiple integrals

Topics

1. Review of Integration
2. Differentiation under the Integral sign
   - Definite integral
   - Indefinite integral
3. Basic Integration Formulas
4. Integral Techniques
   - Integration by parts
   - Integration by change of variable
   - Integration by partial fractions
5. Leibniz's formula

Review of Integration

- An integral is basically the exact opposite of a derivative or the reverse of differentiation
- Also, an integral on the function gives the area under a curve from the x -axis to the curve from a specified range.
- An integral is expressed by the symbol: $\int$

Review of Integration

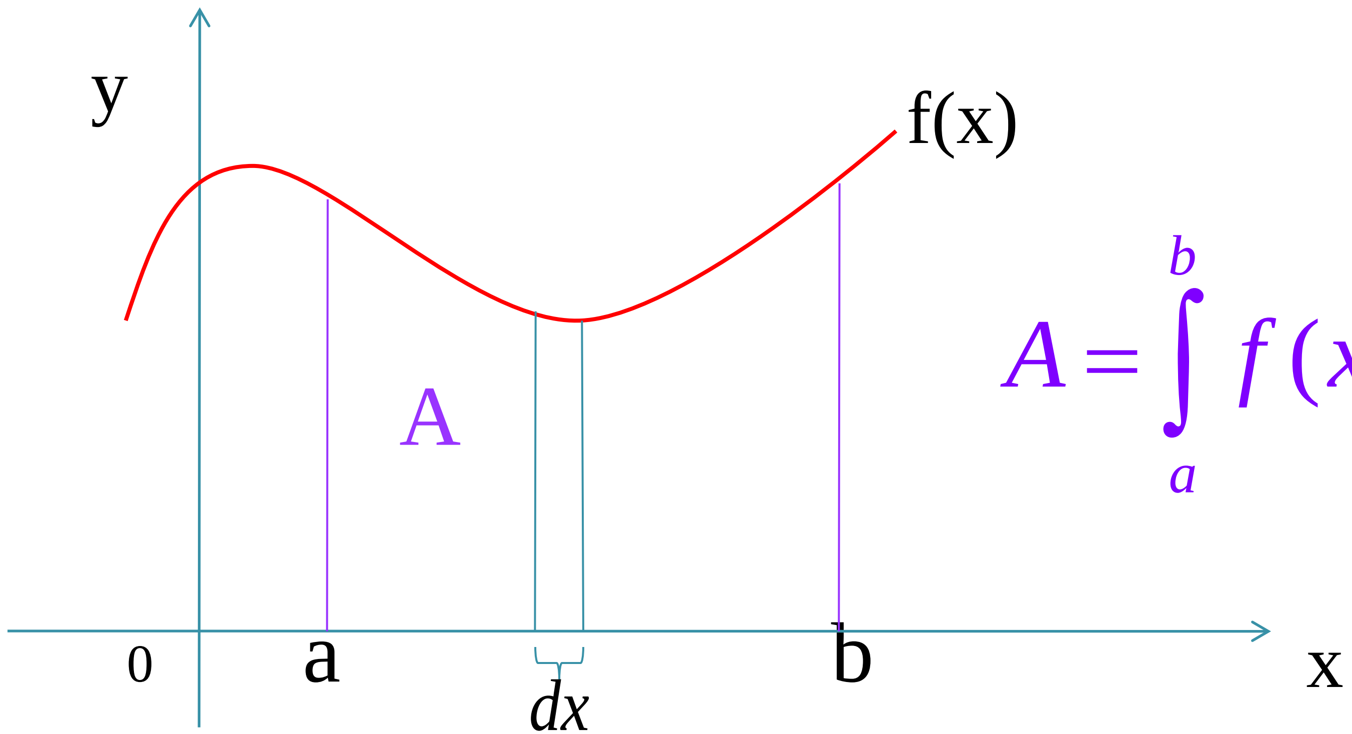
The symbol: $\int_a^b f(x)dx$

means “area of the set situated between the graph of *f*, the x-axis, the line of equation $x=a$ and the line of equation $x=b$”.

- In other words, the area bounded by the curve f, x-axis, the vertical lines x=a and x=b.
- These areas can be approximated by a set of rectangles, or = sum $f(x_i)\ Dx_i$ where Dx is x subinterval. Now, if the length of these subintervals tends to zero, we approach the area under the function.

Review of Integration

- 2 types of area under the curve

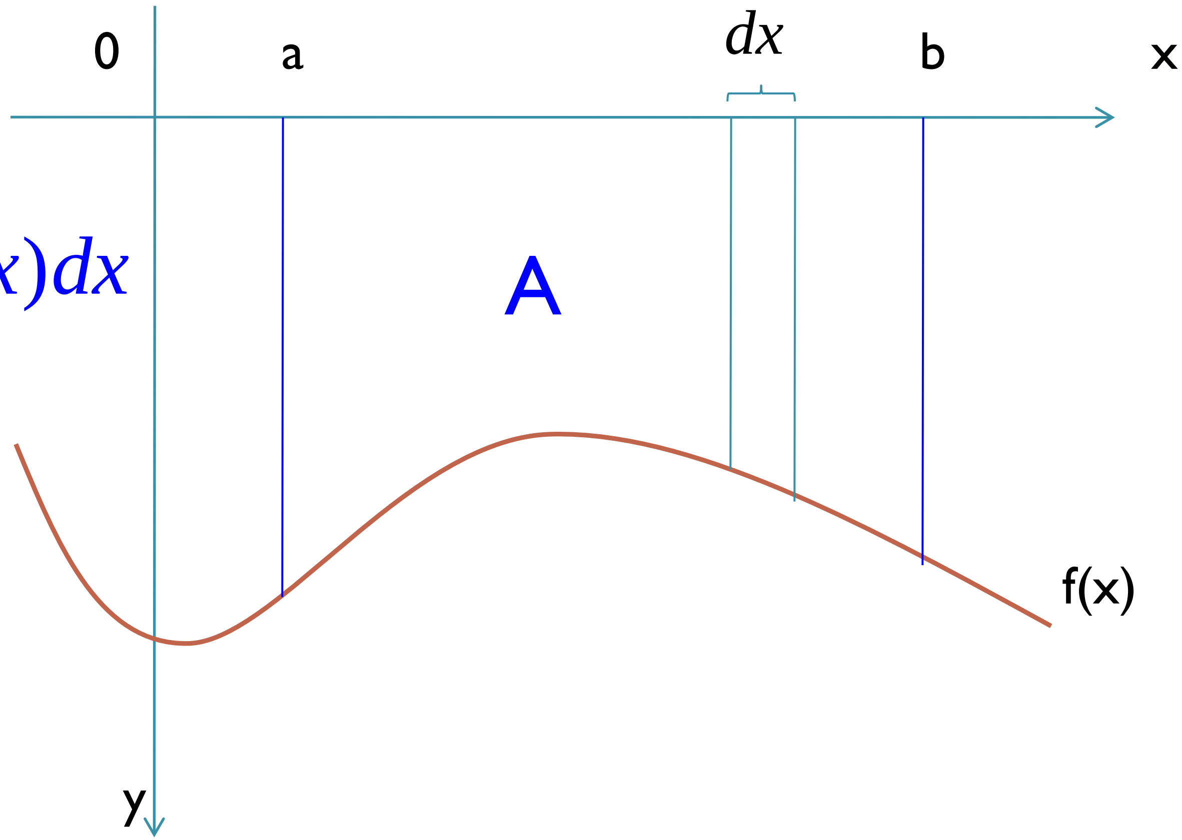


$$A = \int_a^b f(x)dx$$

f(x) > 0

Review of Integration

$$A = -\int_a^b f(x)dx$$



f(x)<0

Differentiation under the Integral sign

- Integration is, intuitively, the opposite of differentiation. Mathematically, we say that if there exists a function such that

$$\frac{d}{dx}F(x) = f(x) \text{ then } \int_a^b f(x)dx = \Big|_a^b F(x) = F(b) - F(a)$$

for all x in (a, b)

The result above is called "**definite integral**"

Differentiation under the Integral sign

- If such a function f exist, then instead of saying "a function F such that

$$\frac{d}{dx}F(x) = f(x)$$ ", we usually say

$$F(x) = \int f(x)dx$$

for all x on an interval ***I*.**

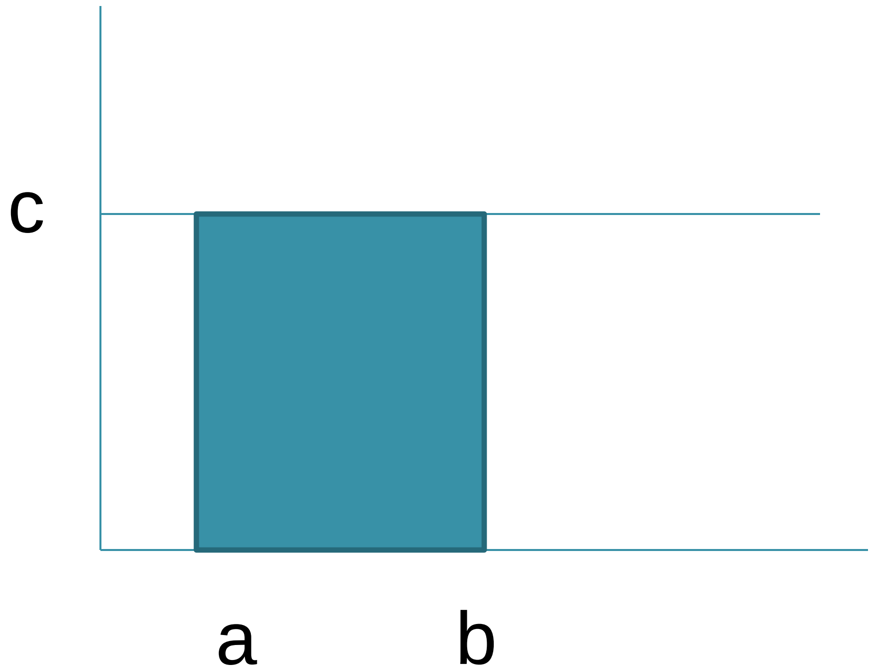
The result above is called "**indefinite integral**"

Basic Integration Formulas

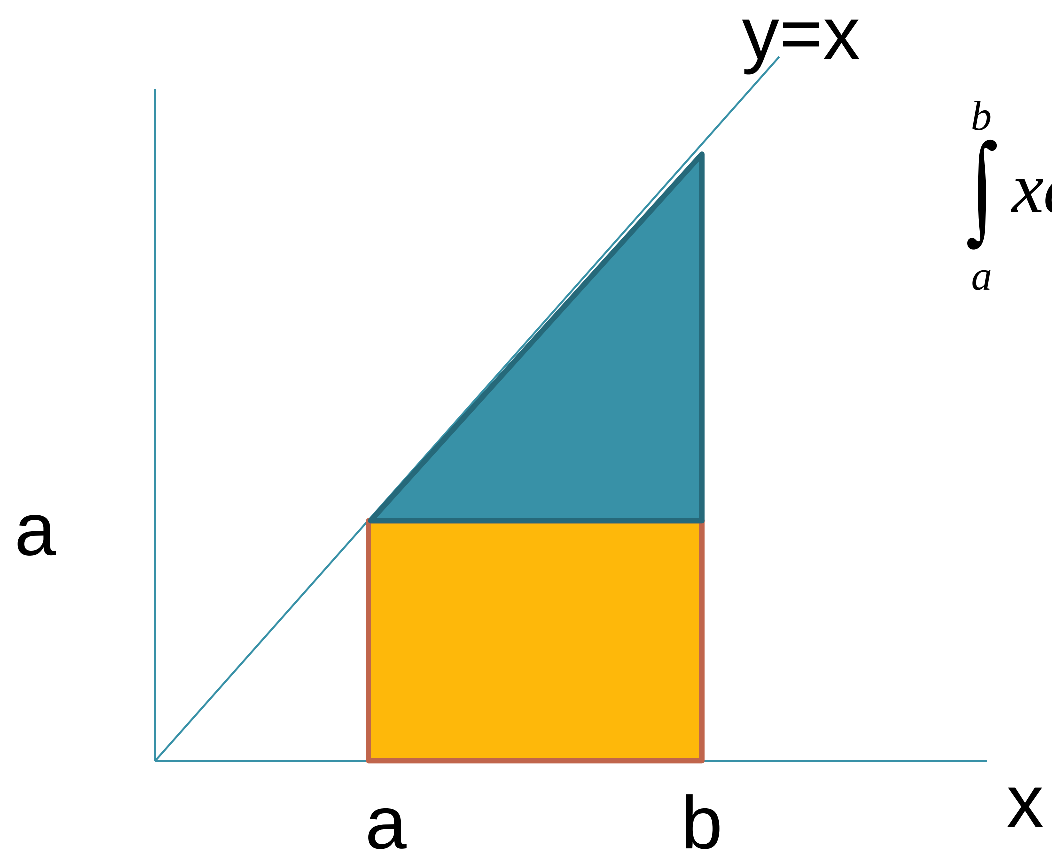
- The basic rules of integration are presented here along with several examples.
- Let *a*, *b*, *n* and *C* are constants,
- *f*(x), *g*(x) are continuous function and differentiable.

1. $\int a\,dx = ax + C$

2. $\int a^x dx = \dfrac{a^x}{\ln a} + C, \qquad a > 0$



$$\int_a^b c\,dx = c(b-a)$$



$$\int_a^b x\,dx = 0.5(b^2 - a^2)$$

Basic Integration Formulas

3. $\int (af(x) \pm bg(x))dx = a\int f(x)dx \pm b\int g(x)dx$

4. $\int ax^n dx = a\int x^n dx = a \cdot \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$

5. $\int \frac{1}{ax} dx = \frac{1}{a} \ln|x| + C$

6. $\int e^{ax} dx = \frac{1}{a} e^{ax} + C \quad (a \neq 0)$

7. $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$

Basic Integration Formulas

8. $\int xe^x dx = xe^x - e^x + C$

9. $\int x^n e^{ax} dx = \frac{x^n}{a} e^{ax} - \frac{n}{a} \int x^{n-1} e^{ax} dx, \qquad a \neq 0$

10. $\int \log_a x dx = x \log_a x - x \log_a e + C, \qquad a > 0$

11. $\int \ln x dx = x \ln x - x + C$

12. $\int x^n \ln x dx = \frac{x^{n+1}\left((n+1)\ln x - 1\right)}{(n+1)^2} + C, n \neq -1$

Basic Integration Formulas

13. $\int \sin x dx = -\cos x + C$

14. $\int \cos x dx = \sin x + C$

15. $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C, \qquad a \neq 0$

16. $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a} \right) + C, \qquad a \neq 0$

17. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a} \right) + C, \qquad a \neq 0$

Basic Integration Formulas

18. $$\int \frac{1}{\sqrt{x^2 \pm a^2}}\,dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$$

19. $$\int \sqrt{x^2 \pm a^2}\,dx = \frac{x}{2}\sqrt{x^2 \pm a^2} \pm \frac{a^2}{2}\ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$$

20. $$\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C, \quad a \neq 0$$

Integration by parts

- From formula 7.

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

- In order to understand this technique, recall the formula

$$\frac{d\left(u(x)v(x)\right)}{dx} = u(x)v'(x) + u'(x)v(x)$$

which implies

$$u(x)v(x) = \int u(x)v'(x)dx + \int u'(x)v(x)dx$$

Integration by parts

- Therefore if one of the two integrals

$$\int u(x)v'(x)dx \quad \text{and} \quad \int u'(x)v(x)dx$$

is easy to evaluate, we can use it to get the other one. This is the main idea behind Integration by Parts.

- Let us give the practical steps how to perform this technique:

Integration by parts

1. Write the given integral

$$\int f(x)g(x)dx$$

where you identify the two functions *f*(*x*) and *g*(*x*).

Note that: *if you are given only one function, then set the second one to be the constant function g(x)=1.*

Integration by parts

2. Introduce the intermediary functions $u(x)$ and $v(x)$ as:

$$u = f(x), \quad dv = g(x)dx$$

- Then you need to make one derivative (of $f(x)$) and one integration (of $g(x)$) to get

$$du = f'(x)dx, \quad v = \int g(x)dx$$

- Note that at this step, you have the choice whether to differentiate f(x) or g(x).

Integration by parts

3. Use the formulas

$$\int u(x)dv = u(x)v(x) - \int v(x)du$$

Take care of the new integral $\int v(x)du$

if the new integral (you will be handling) is easier than the initial one, then your choice was a good one, otherwise go back to Step 2 and make the switch. It is after many integrals that you will start to have a feeling for the right choice.

Integration by parts

- The following examples illustrate the most common cases in which you will be required to use Integration by Parts:

Integration by parts

Ex.1 $\int x^2 e^x dx$

In this case,

$$f(x) = x^2 \text{ and } \quad g(x) = e^x$$

$$\begin{aligned} so, \quad \int x^2 e^x dx &= x^2 e^x - \int e^x d\left(x^2\right) \\ &= x^2 e^x - 2\int x e^x dx \\ &= x^2 e^x - 2\left(x e^x - \int e^x dx\right) \\ &= x^2 e^x - 2\left(x e^x - e^x\right) + C \\ &= x^2 e^x - 2x e^x + 2e^x + C \qquad \# \end{aligned}$$

Integration by Change of Variables

- One of the most important methods of integration is integration by substitution:

$$\int f(x)dx = \int f\left(g(u)\right) g'(u)du, \quad x = g(u)$$

$$ex1. \int \frac{\ln x}{x} dx = \int t dt = \frac{t^2}{2} + C$$

$$set\, t = \ln x, dt = dx / x$$

Integration by Change of Variables

Ex.2 $$\int \frac{2x-9}{\sqrt{x^2-9x+1}}dx$$

Let $u = x^2 - 9x + 1, \quad du = 2x - 9 \; then$

$$\int \frac{2x-9}{\sqrt{x^2-9x+1}}dx = \int \frac{1}{\sqrt{u}}du$$

$$= \int u^{-1/2} du$$

$$= 2u^{1/2} + C$$

$$= 2\sqrt{x^2-9x+1} + C \qquad \#$$

Integration by partial fraction

- This method is based on the simple concept of adding fractions by getting a common denominator.

- This concept can also be used with functions of x. For example,

Integration by partial fraction

$$\frac{2}{x+1}-\frac{1}{x}=\frac{2}{x+1}\left(\frac{x}{x}\right)-\frac{1}{x}\left(\frac{x+1}{x+1}\right)$$

$$=\frac{2x}{x(x+1)}-\frac{x+1}{x(x+1)}$$

$$=\frac{2x-x-1}{x(x+1)}$$

$$=\frac{x-1}{x^2+x}$$

Integration by partial fraction

so that we can now say that a partial fractions

decomposition for $\dfrac{x-1}{x^2+x}$ is

$$\frac{x-1}{x^2+x} = \frac{2}{x+1} - \frac{1}{x}$$

Integration by partial fraction

- Of course, what we would like to be able to do is find a partial fractions decomposition for a given function. For example, what would be a partial fractions decomposition for $\dfrac{6}{x^2 - 1}$?

Integration by partial fraction

- Begin by factoring the denominator, getting

$$\frac{6}{x^2-1}=\frac{6}{(x-1)(x+1)}$$

- Now assume that there are constants A and B so that

$$\frac{6}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}$$

- (Get a common denominator and add the fractions.)

Integration by partial fraction

$$\frac{6}{(x-1)(x+1)} = \frac{A}{x-1}\left(\frac{x+1}{x+1}\right) + \frac{B}{x+1}\left(\frac{x-1}{x-1}\right)$$

$$= \frac{A(x+1)}{(x-1)(x+1)} + \frac{B(x-1)}{(x-1)(x+1)}$$

$$= \frac{A(x+1) + B(x-1)}{(x-1)(x+1)}$$

Integration by partial fraction

- Since the fractions in the equation have the same denominators, it follows that their numerators must be equal. Thus,

$$6 = A(x+1) + B(x-1)$$

- The RHS of this equation can be considered a function of x which is equal to 6 for all values of x. In particular, it must also be true for specific values of x. For example, if we choose to let x=1.

Integration by partial fraction

- Let x=1, then $6 = A(1+1) + B(1-1) = 2A$

$$\therefore A = 3$$

- Let x=-1, then $6 = A(-1+1) + B(-1-1) = -2B$

$$\therefore B = -3$$

- We can now say that a partial fractions decomposition for

$$\frac{6}{(x-1)(x+1)} \quad is \quad \frac{3}{x-1} - \frac{3}{x+1}$$

Integration by partial fraction

Ex. 3 $$\int \frac{1}{x^2-4}\,dx = ?$$

Getting $$\int \frac{1}{x^2-4}\,dx = \int \left(\frac{A}{x-2} + \frac{B}{x+2} \right) dx$$

After getting a common denominator, adding fractions, and equating numerators, it follows that

$$A(x+2) + B(x-2) = 1$$

Integration by partial fraction

- Let x=2, then $$1 = A(2+2) + B(2-2)$$ $$\therefore A = 1/4$$
- Let x=(-2), then $$1 = A(-2+2) + B(-2-2)$$ $$\therefore B = -1/4$$

Integration by partial fraction

- So, $$\int \frac{1}{x^2-4}dx = \int\left(\frac{1}{4(x-2)} - \frac{1}{4(x+2)}\right)dx$$

$$= \frac{1}{4}\int \frac{1}{x-2}dx - \frac{1}{4}\int \frac{1}{x+2}dx$$

$$= \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C$$

$$= \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C \qquad \#$$

Leibniz's Formula

- Integrals appearing in economics often depend on parameters.
- How does the value of the integral change if the parameters change?
- Let f be a function of two variables and consider the function of F defined by

$$F(x) = \int_a^b f(x,t)dt$$

where a and b are constants.

- we want to find $F'(x)$

- Since the limits of integration don't depend on x, it is natural to guess that we have the following result:
- $$F(x)=\int_a^b f(x,t)dt \Rightarrow F'(x)=\int_a^b \frac{\partial f(x,t)}{\partial x}dt$$
- Thus we differentiate the integral w.r.t. a parameter that occurs only under the integral sign, by differentiating under the integral sign.

Ex.

- $K = \int_0^T f(t)e^{-rt}dt, \quad find \frac{dK}{dr}.$
- $\frac{dK}{dr} = \int_0^T f(t)(-t)e^{-rt}dt$
- $\quad = -\int_0^T f(t)te^{-rt}dt$

Theorem 4.2.1 Leibniz's Formula

Suppose that $f(x,t)$ and $f_x'(x,t)$ are continuous over $a \le x \le b, c \le t \le d$. Suppose u(x) and v(x) are C^1 over [*a*, *b*], and the range of *u* and *v* are contained [*c*, *d*]. Then

$$F(x) = \int_{u(x)}^{v(x)} f(x,t)\,dt$$

$$\Rightarrow \quad F'(x) = f\left(x, v(x)\right)v'(x) - f\left(x, u(x)\right)u'(x) + \int_{u(x)}^{v(x)} \frac{\partial f(x,t)}{\partial x}dt$$

Leibniz's Formula

Ex.4 Compute $F'(x)$ when $F(x)=\int_{x}^{x^2}\frac{1}{2}t^2 x\,dt$.

$$F'(x)=\frac{1}{2}\left(x^2\right)^2 x\cdot 2x-\frac{1}{2}x^2 x\cdot 1+\int_{x}^{x^2}\frac{1}{2}t^2 dt$$

$$=x^6-\frac{1}{2}x^3+\left|_{x}^{x^2}\frac{1}{6}t^3\right.$$

$$=x^6-\frac{1}{2}x^3+\frac{1}{6}\left(x^6-x^3\right)$$

$$=\frac{7}{6}x^6-\frac{2}{3}x^3$$

Leibniz's Formula

or $$F(x) = \frac{1}{2} x \int_{x}^{x^2} t^2 dt = \frac{1}{2} x \left| \begin{matrix} x^2 \\ x \end{matrix} \frac{1}{3} t^3 \right. = \frac{1}{6}\left(x^7 - x^4\right)$$

Diff w.r.t. x,

$$F'(x) = \frac{7}{6} x^6 - \frac{4}{6} x^3 = \frac{7}{6} x^6 - \frac{2}{3} x^3 \qquad \#$$

Ex. Leibniz's Formula

- Let y(t) is a net profit stream between [0,T]
- Suppose *r* is a constant discount rate
- At time s, the discount value of profit is
- $v(r,s) = \int_s^T y(t)e^{-r(t-s)}dt$
- Find dv/ds
- $\frac{\partial v(r,s)}{\partial s} = -y(s) + \int_s^T y(t)re^{-r(t-s)}dt$
- $= -y(s) + rV(r,s)$