Assignment 4

DUE DATE: Tuesday 9^{nd}, March 2021.

I pledge to the Honor Code and to obey all rules for taking and performing homework assignments as specified by the course instructor.

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Question 1 (50 points) Your score......................

Given the daily log returns : (R_t) can be explained by the AR(2) model as following:

$$(1-1.5B+0.9B^2)R_t = 0.25+\varepsilon_t$$

where ε_t is distributed as the Gaussian White Noise with mean (μ) = 0 and variance (σ^2) =0.25
B lag-operator

Question 1.1 (10 points) Your score......................

From the above AR(2) model, Is the model weakly stationary? Write down the reverse characteristic equation and find out the conditions to support your answer.

$(1-1.5B+0.9B^2)R_t = 0.25+\varepsilon_t$

$\downarrow$

For reverse characteristic equation : $x^2 - 1.5x + 0.9$

$\rightarrow$ The condition for AR(2) to have weak stationarity are 1) $E(R_t)$ 2) $Var(R_t)$ 3) $Cov(R_t, R_{t-j})$ are constant.

Condition also
$|\phi_1| < 0$

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where ε_t is distributed as the Gaussian White Noise with mean $(\mu)=0$ and variance (σ^2) =0.25
B lag-operator

Question 1.2 (10 points)

Your score......................

Calculate the unconditional mean: $E(R_t)$ of R_t and the conditional mean: $E(R_t|F_{t-1})$

For unconditional mean :

rewrite equation as $R_t - 1.5R_{t-1} + 0.9R_{t-2} = 0.25 + \varepsilon_t$

$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t$

take $E[\cdot] \Rightarrow E[R_t] = 0.25 + 1.5E[R_{t-1}] - 0.9E[R_{t-2}] + E[\cancel{\varepsilon_t}]^{0}$

assumption for weakly stationarity $\Rightarrow E[R_t] = E[R_{t-h}]$

$E[R_t] - 1.5E[R_t] + 0.9E[R_t] = 0.25$

$E[R_t] = \dfrac{0.25}{1-1.5+0.9} = 0.625$

For conditional mean :

take $E[r_t \mid \cdot\,] \Rightarrow E[R_t \mid F_{t-2}] = 0.25 + 1.5E[R_{t-1}|\cdot] - 0.9E[R_{t-2}|\cdot] + E[\cancel{\varepsilon_t}^{0}|\cdot]$

$\hat{R}_t(2) = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + 0$

Question 1.3 (10 points) | Your score......................

Find out the unconditional variance: $Var(R_t)$ of R_t and conditional variance $Var(R_t|F_{t-1})$ of R_t

$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t$

start with deviation form : $R_t - \mu = 1.5(R_{t-1} - \mu) - 0.9(R_{t-2} - \mu) + a_t$

then take squared $\Rightarrow (R_t - \mu)^2 = 1.5^2(R_{t-1} - \mu)^2 + 0.9^2(R_{t-2} - \mu)^2 + a_t^2 + 2(1.5)(0.9)(R_{t-1} - \mu)(R_{t-2} - \mu)$

$+ 2(1.5)(R_{t-1} - \mu)\, a_t$

$+ 2(0.9)(R_{t-2} - \mu)\, a_t$

$\sigma^2 = 0.25$

now take $E[\cdot] \Rightarrow E[(R_t - \mu)^2] = (1.5)^2 E[(R_{t-1} - \mu)^2] + (0.9)^2 E[(R_{t-2} - \mu)^2] + E[a_t^2] + 2(1.5)(0.9) E[(R_{t-1} - \mu)(R_{t-2} - \mu)]$

$+ 2(1.5)(0) + 2(0.9)(0)$

$Var(R_t) = (1.5)^2 Var(R_{t-1}) + (0.9)^2 Var(R_{t-2}) + 2(1.5)(0.9)(\gamma_1)$ $+0.25$

assume weakly stationary : $Var(R_t) - (1.5)^2 Var(R_t) - (0.9)^2 Var(R_t) = 2(1.5)(1.9)(\gamma_1) + 0.25$

$$Var(R_t) = \frac{2(1.5)(0.9)(\gamma_{t-1}) + 0.25}{\underbrace{1 - (1.5)^2 - (0.9)^2}_{(-2.06)}} = -1.3107\gamma_{t-1} - 0.1214$$

For conditional var. : a/f take ^{2} to deviation form

we take $E[\cdot \mid F_{t-2}] \Rightarrow E[(R_t - \mu)^2 \mid \cdot] = (1.5)^2 E[(R_{t-1} - \mu)^2 \mid \cdot] + 0.9^2 E[(R_{t-2} - \mu)^2 \mid \cdot] + E[\cancel{a_t^2} \mid \cdot] + 2(1.5)(0.9) E[(R_{t-1} - \mu)(R_{t-2} - \mu) \mid \cdot]$

$+ 0 + 0$

$\therefore Var(R_t \mid \cdot) = (1.5)^2 Var(R_{t-1} \mid \cdot) + 0.9^2 Var(R_{t-2} \mid \cdot) + 2(1.5)(0.9) cov(R_{t-1}, R_{t-2} \mid \cdot$

Question 1.4 (10 points) Your score......................

note $\phi_1 = 1.5$ and $\phi_2 = 0.9$

Calculate the autocorrelation: ρ_l for l=1 and 2 of R_t. Also, write down the autocorrelation: ρ_l when l $\geq$ 2.

ρ_l for $\rho_{l=1}$ & $\rho_{l=2}$ and $\rho_{l>2}$

Start w/ deviation form.

for ρ_l : $R_t - \mu = 1.5(R_{t-1} - \mu) - 0.9(R_{t-2} - \mu) + a_t$

take $(R_{t-l} - \mu)$ to both then

$$(R_t - \mu)(R_{t-l} - \mu) = 1.5(R_{t-1} - \mu)(R_{t-l} - \mu) - 0.9(R_{t-2} - \mu)(R_{t-l} - \mu) + (a_t)(R_{t-l} - \mu)$$

take $E(\cdot)$ both :

$$\gamma_l = 1.5\gamma_{l-1} - 0.9\gamma_{l-2} + 0$$

then as $\boxed{\rho_l = \frac{\gamma_l}{\gamma_0}}$ $\Rightarrow$ $\rho_1 = \frac{(1.5\gamma_0 - 0.9\gamma_{-1})}{\gamma_0} = 1.5 - \frac{0.9\gamma_{-1}}{\gamma_0} = \boxed{1.5 - 0.9\rho_{-1}}$ $\rightarrow$ let $\rho_{-1} = \rho_1$ then $\rho_1 = 1.5 - 0.9\rho_1$

$\rho_1 = \frac{1.5}{1+0.9} = \frac{\phi_1}{1-\phi_2}$ $= 0.7895$

$\boxed{\rho_0 = 1}$

$\rho_2 = \frac{(1.5\gamma_1 - 0.9\gamma_0)}{\gamma_0} = 1.5\rho_1 - 0.9 = \frac{1.5(1.5)}{1+0.9} - 0.9 = \frac{\phi_1^2}{1-\phi_2} + \phi_2 = \frac{\phi_1^2 + \phi_2 - \phi_2^2}{1-\phi_2} = \frac{\phi_1^2 + \phi_2(1-\phi_2)}{(1-\phi_2)}$

$= 2.0842$

ρ_l for $l \geq 2$

Question 1.5 (10 points) Your score......................

Given $R_{1000} = 0.01$ $R_{999} = 0.02$ $R_{998} = 0.03$ $\varepsilon_{1000} = -0.01$ $\varepsilon_{999} = -0.02$ $\varepsilon_{998} = -0.03$ Obtain 1-step, 2-setp 95 % interval forecasts for R_t at the forecast origin t = 1000. Also the ∞-step 95 % interval forecasts for R_t. Draw these intervals.

$R_t = 0.25 + 1.5R_{t-1} - 0.9R_{t-2} + \varepsilon_t$

now 1-step forcast ⇒ $R_{h+1} = 0.25 + 1.5R_h - 0.9R_{h-1} + \varepsilon_{h+1}$ $(h = t)$

① ∴ take $E[\cdot|F_h]$ ⇒ $\hat{R}_t(1) = 0.25 + 1.5E[R_{1000}|\cdot] - 0.9E[R_{999}|\cdot] + E[\varepsilon_{1001}|\cdot]$

$\hat{R}_t(1) = 0.25 + (1.5)(0.01) - (0.9)(0.02) + 0$

$\hat{R}_t(1) = \boxed{0.247}$ #

② 1-step forcasting error ⇒ $R_{t+1} - \hat{R}_t(1) = e_t(1) = a_{t+1} = \varepsilon_{t+1}$

③ Var of error ⇒ $var(a_{t+1}|\cdot) = \sigma_a^2$

④ interval forcasting ⇒ $\hat{R}_t(1) \pm 1.96\sqrt{\sigma_a^2} = \boxed{\hat{R}_t(1) \pm 1.96\sigma_a}$

$\boxed{\text{as } 95\%}$

$= 0.247 \pm 1.96\sigma_a$

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① $\hat{R}_t(2) = 0.25 + 1.5E[R_{t+1}|\cdot] - 0.9R_t + \cancel{\varepsilon_{t+1}}^{0}$ vs. $R_{t+2} = \overset{(0.25)}{\phi_0} + \overset{(1.5)}{\phi_1} R_{t+1} + \overset{(-0.9)}{\phi_2} R_t + \varepsilon_{t+2}$

$\hat{R}_t(2) = 0.25 - 0.9(0.01) + 1.5E[R_{t+1}|\cdot]$

$\hat{R}_t(2) = 0.241 + 1.5E[R_{t+1}|\cdot] = 0.241 + 1.5(0.247) = 0.6115$

② 2-step forcasting error : $R_{t+2} - \hat{R}_t(2) = 1.5\underbrace{(R_{t+1} - \hat{R}_t(1))}^{\varepsilon_{t+1}} + \varepsilon_{t+2} = \boxed{1.5\varepsilon_{t+1} + \varepsilon_{t+2}}$

③ Var of forcasting error : $var(1.5\varepsilon_{t+1} + \varepsilon_{t+2}) = (1.5)^2\sigma^2 + \sigma^2 + \cancel{2cov[1.5\varepsilon_{t+1}, \varepsilon_{t+2}|\cdot]}^{0}$

$= \boxed{3.25\sigma^2}$

④ Interval : $\hat{R}_t(2) \pm 1.96\sqrt{3.25\sigma^2}$

$\boxed{= 0.6115 \pm 1.96(1.803)\sigma}$