Practice problem set 5 (Solution)

EE320 Semester 1/2016

Chapter 7: Multivariate calculus: basis derivative and applications

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Question 1: Determine Z_x, $and\ Z_y$

a. $Z = (x^2y+2)^2$

$Z_x = 2(x^2y+2)(2xy)$

$Z_y = 2(x^2y+2)(x^2)$

b. $Z = x^{1/2}y^{1/3}$

$Z_x = (1/2)x^{-1/2}y^{1/3}$

$Z_y = (1/3)x^{1/2}y^{-2/3}$

c. $Z = xy - ln(xy)$

$Z_x = y-(1/x)$

$Z_y = x-(1/y)$

Question 2:

Consider a function $Z = f(x,y) = x^3+3x^2y+6xy^2-y^3$

a. Derive the Hessian matrix.

$$H = \begin{bmatrix} 6x+6y & 6x+12y \\ 6x+12y & 12x-6y \end{bmatrix}$$

b. Evaluate the value of the Hessian matrix where x = 2 and y =3

$$H = \begin{bmatrix} 30 & 48 \\ 48 & 6 \end{bmatrix}$$

Question 3:

a. Given that $Z = \frac{x^3-y^3}{x^2y^2}$, show that $x\frac{\partial Z}{\partial x} + y\frac{\partial Z}{\partial y} = -Z$

$$Z_x = \frac{x^2y^2(3x^2)-(x^3-y^3)(2xy^2)}{x^4y^4} \rightarrow$$

$$xZ_x = \frac{x^3y^2(3x^2)-(x^3-y^3)(2x^2y^2)}{x^4y^4}$$

$$Z_y = \frac{x^2y^2(-3y^2)-(x^3-y^3)(2x^2y)}{x^4y^4} \rightarrow$$

$$yZ_y = \frac{x^2y^2(-3y^3)-(x^3-y^3)(2x^2y^2)}{x^4y^4}$$

$$xZ_x + yZ_y = -\frac{(x^3-y^3)(x^2y^2)}{x^4y^4} = -\frac{(x^3-y^3)}{x^2y^2} = -Z$$

b. Given that $Z = \frac{x-y}{x+y}$, find $x\frac{\partial Z}{\partial x} + y\frac{\partial Z}{\partial y} = ?$

$$Z_x = \frac{2y}{(x+y)^2} \rightarrow xZ_x = \frac{2xy}{(x+y)^2}$$

$$Z_y = \frac{-2x}{(x+y)^2} \rightarrow yZ_y = \frac{-2xy}{(x+y)^2}$$

$$xZ_x + yZ_y = 0 = 0 * Z$$

Question 4

Suppose that the demand of a product is given by,

$$Q_x = 100 - 4P_x - \frac{50}{\sqrt{P_y}} + 0.5I^2 .$$

where P_x is the price of good X, P_y is the price of good Y, and I is the level of income. Consider the following problems

a. What is the relationship between good x and good y? Are they substitute product/complementary product? Show your results using the partial derivative

$\partial Q_x/\partial p_y = \frac{25}{\sqrt{P_y^3}} > 0$

An increase in price of y causes an increase in quantity of X. Thus, x and y are substitute product.

b. Is the product X considered an inferior product?

$\partial Q_x / \partial I \;=\; I > 0.$

X is an normal good. So, No, x is not an inferior product.

c. What is the level of quantity demanded if $P_x = 10,\ P_y = 25\ \ and\, I\ =\ 10$?

$Q_x = 100 - 40 - 10 + 50\ =\ 100$

d. Calculate the own-price elasticity of demand, and evaluate the value when $P_x = 10,\ P_y = 25\ \ and\, I\ =\ 10$.

From c, we know that $Q_x = 100$

$\varepsilon_{x,px} = \frac{\partial Q_x}{\partial P_x} * \frac{P_x}{Q_x} = (-4) * \frac{10}{100} = -0.4$

e. Calculate the cross-price elasticity of demand when $P_x = 10,\ P_y = 25\ \ and\, I\ =\ 10$.

$\varepsilon_{x,py} = \frac{\partial Q_x}{\partial P_y} * \frac{P_y}{Q_x} = \frac{25}{\sqrt{P_y^{\,3}}} * \frac{25}{100} = \sqrt{25}/100$

f. Calculate income elasticity of demand when $P_x = 10,\ P_y = 25\ \ and\, I\ =\ 10$. Is the product a necessary or luxurious product?

$\varepsilon_{x,I} = \frac{\partial Q_x}{\partial I} * \frac{I_x}{Q_x} = (10)\frac{10}{100} =\ 1$

Normally, if income elasticity is greater than one, the product is classified as luxurious goods. On the other hand, if income elasticity is less than one, the production would be classified as necessary goods. For the case with unit elasticity, it doesn't have a specific name to which people usually refer.

Question 5: Consider a simple market model where

Demand: $q\ =\ -p + \sqrt{I}$

Supply: $q = 3p\ -\ w^2$

where q is quantity of output, p is price, I is the level of income, and w is the price of factor input

a) Derive the solution of all the endogenous variables.

$p^* = 0.25(\sqrt{I} + w^2)$

$q^* = (0.75\sqrt{I} - 0.25w^2)$

b) Use the partial derivative to show that, under the equilibrium, an increase in W causes a decrease in quantity, but an increase in price.

$\partial p^*/\partial w = \frac{1}{2}w > 0 \qquad \partial q^*/\partial w = -\frac{1}{2}w < 0$

c) Predict the impact of an increase in income on equilibrium quantity and price.

Note that $\partial p^*/\partial I = \frac{1}{8\sqrt{7}} > 0 \qquad \partial q^*/\partial I = \frac{3}{8\sqrt{7}} > 0$.
Both are positive. Thus, an increase income would give rise to an increase in both price and quantity.

Question 6: Find the total differentials for the following functions:

a. $z = 4x^3 - 13xy - 6y^5$

Ans. $dz = (12x^2 - 13y)dx + (-13x - 30y^4)dy$

b. $z = (2x^2 - y)(3x - 4y^3)$

Ans. $dz = (18x^2 - 16xy^3 - 3y)dx + (-24x^2y^2 + 16y^3 - 3x)dy$

c. $z = 8x^{\frac{1}{2}}y^{\frac{1}{4}}w^{\frac{1}{4}}$

Ans. $dz = (4x^{-1/2}y^{1/4}w^{1/4})dx + (2x^{1/2}y^{-3/4}w^{1/4})dy + (2x^{1/2}y^{1/4}w^{-3/4})dw$

Question 7: Suppose that Mr. A's utility depends on two commodities: x_1 and x_2. His utility function is given by

$$U = 25x_1 + 27x_2 - 3x_1^2 - 7x_1x_2 - 4x_2^2$$

(a) Determine the marginal utility function of each commodity.

Ans. $MU_1 = 25 - 6x_1 - 7x_2$

$MU_2 = 27 - 7x_1 - 8x_2$

(b) Find the marginal rate of substitution (MRS) between the two commodities when $x_1 = 2$ and $x_2 = 1$.

Ans. $MRS_{12} = \frac{dx_2}{dx_1} = -\frac{MU_1}{MU_2} = -1.2$

Question 8: Given the equation for the production isoquant

$$18[0.2K^{-0.4} + 0.8L^{0.4}]^{-2.5} = 1936$$

Use the implicit function rule to find the marginal rate of technical substitution (MRTS) of L for K.

Ans.

Let $F(K,L) \equiv 18[0.2K^{-0.4} + 0.8L^{0.4}]^{-2.5} - 1936 = 0$

Use implicit function rule: $\frac{dK}{dL} = -\frac{F_L}{F_K}$

$F_L = 14.4[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5} \times L^{-1.4}$

$F_K = 3.6[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5} \times K^{-1.4}$

$\Rightarrow \; MRTS = \frac{dK}{dL} = -\frac{F_L}{F_K} = -4(\frac{K}{L})^{1.4}$

Question 9: The demand for money, *M*, in the United States for the period 1929-1952 has been estimated as

$$M = 0.14Y + 76.03(r-2)^{-0.84} \quad , \qquad (r > 2)$$

where *Y* is the annual national income, and *r* is the interest rate measured in percent per year. Find $\frac{\partial M}{\partial Y}$ and $\frac{\partial M}{\partial r}$ and discuss their signs.

Ans. $\partial M/\partial Y = 0.14 > 0$: Money demand increases with income.

$\partial M/\partial r = -0.84 \cdot 76.03(r-2)^{-1.84} = -63.87(r-2)^{-1.84} < 0$: Money demand decreases as the interest rate increases.

Question 10: Suppose that a firm produces $Q = f(L) = L^{1/2}$ units of commodity using *L* units of labor. Assume that *f'(L) > 0* and *f''(L) < 0*, so *f* is strictly increasing and strictly concave. If the firm gets *P* baht per unit produced and pays *w* baht for a unit of labor, write down the profit function, and find the first-order condition for profit maximization at *L* > 0*.

Ans. Profit function: $(L) = Pf(L) - wL$.

F.O.C. $Pf'(L) - w = 0 \Rightarrow \frac{P}{2\sqrt{L^*}} = w \Rightarrow L^* = \frac{P^2}{4w^2}$.

***Question 11**: Let the demand for a commodity be

$$Q_d = D(P, Y_0, T_0) \, , \qquad \frac{\partial D}{\partial P} < 0; \frac{\partial D}{\partial Y_0} > 0; \frac{\partial D}{\partial t_0} < 0$$

$$Q_s = S(P, T_0), \qquad \frac{\partial S}{\partial P} > 0; \frac{\partial S}{\partial T_0} < 0$$

Where P is the price, Y_0 is the consumer's income, t_0 is the taste for the commodity and T_0 is the tax on the commodity. All derivatives are continuous. Use implicit function rule to find $\frac{\partial P^*}{\partial t_0}$, $\frac{\partial P^*}{\partial T_0}$, $\frac{\partial Q^*}{\partial Y_0}$, and $\frac{\partial Q^*}{\partial T_0}$. Discuss their economic implications.

Ans. Let P^* be the equilibrium price: $P^* = P^*(Y_0, t_0, T_0)$.

At equilibrium, $D(P^*, Y_0, t_0) = S(P^*, T_0)$

Implicit function: $F \equiv D(P^*, Y_0, t_0) - S(P^*, T_0) = 0$.

Using implicit function rule,

$\Rightarrow \frac{\partial P^*}{\partial t_0} = -\frac{\partial F_{t_0}}{\partial F_{P^*}} = -\frac{\partial D/\partial t_0}{\partial D/\partial P^* - \partial S/\partial P^*} > 0$

(As the taste increases, the equilibrium price rises.)

$\Rightarrow \frac{\partial P^*}{\partial T_0} = -\frac{\partial F_{T_0}}{\partial F_{P^*}} = -\frac{-\partial S/\partial T_0}{\partial D/\partial P^* - \partial S/\partial P^*} > 0$

(As the tax imposed on producer increases, the equilibrium price increases.)

$\Rightarrow \frac{\partial Q^*}{\partial Y_0} = \frac{\partial S}{\partial P^*}\frac{\partial P^*}{\partial Y_0}$ since $Q^* = S(P^*, T_0) = D(P^*, Y_0, t_0)$ and $P^* = P^*(Y_0, t_0, T_0)$

$\frac{\partial P^*}{\partial Y_0} = -\frac{\partial F_{Y_0}}{\partial F_{P^*}} = -\frac{-\partial D/\partial Y_0}{\partial D/\partial P^* - \partial S/\partial P^*} > 0$

$\Rightarrow$ Since $\frac{\partial S}{\partial P^*} > 0$ and $\frac{\partial P^*}{\partial Y_0} > 0$, $\frac{\partial Q^*}{\partial Y_0} = \frac{\partial S}{\partial P^*}\frac{\partial P^*}{\partial Y_0} > 0$

(As income rises, equilibrium quantity increases.)

$\Rightarrow \frac{\partial Q^*}{\partial T_0} = \frac{\partial S}{\partial T_0} < 0$ since $Q^* = S(P^*, T_0)$

(As tax increases, equilibrium quantity decreases.)

Question 12:

A joint-cost function is defined implicitly by the equation

$$c + \sqrt{c} = 12 + q_A\sqrt{9 + q^2_B}$$

where c is denoted as the total cost for producing q_A and q_B units of product A and B, respectively.

a) If $q_A = 6$ and $q_B = 4$, find the corresponding value of c. **Ans. c = 36**

b) Determine the marginal costs with respect to q_A and q_B, when $q_A = 6$ and $q_B = 4$.

MCa = 60/11

MCb = (24*12)/55

Question 13:

Suppose a production function is given by $P = \frac{kl}{k+l}$

a. Determine the marginal productivity functions.
   MPK = $L^2/(K^2+L^2)$ **and MPL =** $K^2/(K^2+L^2)$
b. Show that when $k = l$, the marginal productivities are equal.
   K = L ; both MPK and MPL will be equal to ½

Question 14:

Suppose the demand equations for related products A and B are

$q_A = e^{-(p_A+p_B)}$ and $q_B = \frac{16}{p_A^2\ p_B^2}$

where q_A and q_B are the number of units of A and B demanded when the unit prices (in thousands of dollars) are p_A and p_B, respectively.

a. Classify A and B as competitive, complementary, or neither.

$\partial q_A/\partial p_B = e^{-(p_A+p_B)}(-1) < 0$ –> complementary.

b. Calculate elasticity of demand for good A and good B, respectively. (Get me all whatsoever you can do for the elasticities.)

Elasticity of ?? with respect to ??	Qa	Qb
p_A	$-p_A$	-2
p_B	$-p_B$	-2

c. If the unit prices of A and B are $1000 and $2000, respectively, estimate the change in the demand for A when the price of B is decreased by $20 and the price of A is held constant.
   Ans. $20 * e^{-(3000)}$

Question 15:

Suppose the cost c of producing q_A units of product A and q_B units of product B is given by $c = (3q_A^2 + q_B^3 + 4)^{1/3}$. And the coupled demand functions for the products are given by

$q_A = 10 - p_A + p_B^2$ and $\quad q_B = 20 + p_A - 11p_B$

Use a chain rule to evaluate $\frac{\partial c}{\partial p_A}$ and $\frac{\partial c}{\partial p_B}$ when $p_A = 25$ and $p_B = 4$.

$p_A = 25$ and $p_B = 4. \rightarrow q_A = 1$ and $q_B = 1 \rightarrow c = 2$

$\frac{\partial c}{\partial p_A} = \frac{\partial c}{\partial q_A}\frac{\partial q_A}{\partial p_A} + \frac{\partial c}{\partial q_B}\frac{\partial q_B}{\partial p_A} = (6/4)(-1) + (3/4)(1) = 3/4$

$\frac{\partial c}{\partial p_B} = \frac{\partial c}{\partial q_A}\frac{\partial q_A}{\partial p_B} + \frac{\partial c}{\partial q_B}\frac{\partial q_B}{\partial p_B} = (6/4)(8) + (3/4)(-11) = 15/11$