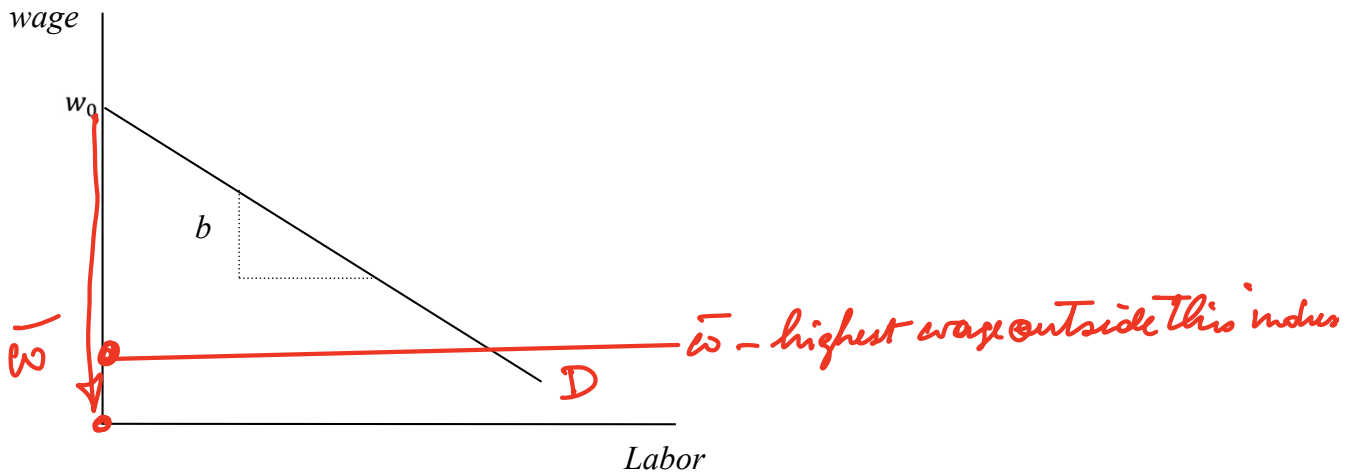


## Chapter 3 Calculus of Single Variable: Applications

**3.1 Labor Union** In a given industry, the labor union is effectively the monopoly of labor input. The market demand for the labor is a negatively sloped as shown in the figure below.



**Figure 3.1** Demand for union workers

The demand for union workers is given by,

$$w = w_0 - bL$$

where  $w$  is the wage earned by union members in a given industry employing  $L$  union workers.

There are 4 possible goals the union can have:

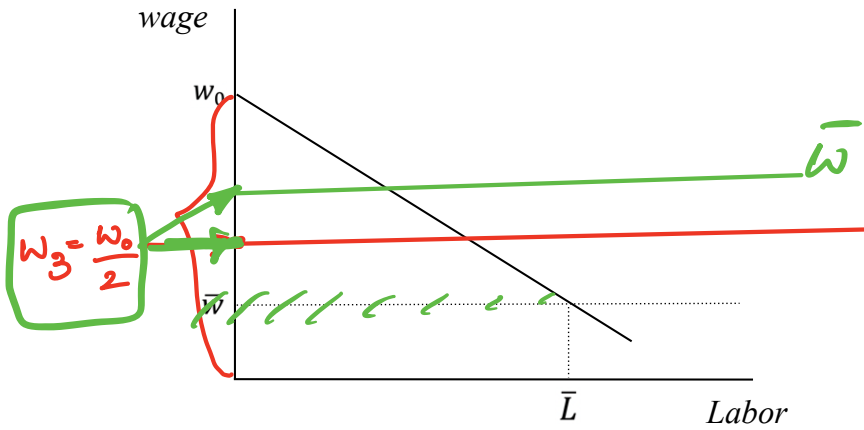
1. max wage rate =  $w_0$ .
  2. max level of employment  $L$
  3. max total income of employed workers
  4. max rents earned by employed workers
- $\max w \cdot L \iff \max x \cdot (wL)$   
 $x > 0.$

1) Max Wage Rate: Optimal solution  $w^* = w_0 = w_1$ , and so  $L = 0$  and it is a corner solution—a solution where the sufficient conditions are not met.

2) Max Employment: Let

$\bar{w}$  = wages earned by union members working in other industry or receiving unemployment compensation.

Then, optimal solution  $w^* = \bar{w} = w_2$ . That is, the union will set the wage at the level the worker can get at the other best paying industry and  $\bar{L}$  members will be hired.



**Figure 3.2** The best wage rate to set if the union maximizes employment

3) Max Total Income of Employed Union Workers:

Let  $I$  be the total income earned by all employed union workers. We can see that the total income depends on wage rate  $w$ , and so we write  $I = I(w)$  where,

$$I(w) = wL = \frac{w(w_0 - w)}{b}, \quad w = w_0 - bL.$$

By the first-order sufficient condition,

$$I'(w^*) = \frac{w_0 - 2w^*}{b} = 0,$$

and we have the critical point  $w^* = \frac{w_0}{2}$ . We can test that this critical point satisfies the second-order sufficient condition

$$I''(w^*) = -\frac{2}{b} < 0.$$

*same for any w.  $I''(w) = -\frac{2}{b}$   
So  $I''(w^*) = -\frac{2}{b} < 0$*

So the optimal wage rate to set is  $w_3 = \max\{\frac{w_0}{2}, \bar{w}\}$ .

4) Max Economic Rent: We define rent as

$$r = \frac{400}{b} - \bar{w}, \quad \text{opp cost} = 300 \text{ B}$$

$$\text{rent} = 400 - 300 = 100 \text{ B}$$

i.e., the difference between the wage received and the highest wage earned in other industry. The total rent earned is

$$R(w) = rL = (w - \bar{w}) \frac{w_0 - w}{b}$$

first-order sufficient condition,

$$R'(w^*) = \frac{\bar{w} + w_0 - 2w^*}{b} = 0 \quad \checkmark$$

and the critical point is

$$w^* = \frac{\bar{w} + w_0}{2} \quad \checkmark$$

The second-order sufficient condition is given by

$$R''(w^*) = -\frac{2}{b} < 0. \quad \checkmark$$

We will have the optimal solution

$$w_4 = \frac{\bar{w} + w_0}{2} = w_3 + \frac{\bar{w}}{2} \text{ if } w_3 > \bar{w}.$$

$w_3 = \frac{\bar{w}}{2}$        $w_4 > w_3.$

- Are  $w_3$  and  $w_4$  strict global maximum points?

In summary, we can draw a graph to show the relation of optimal wage rates of four cases as

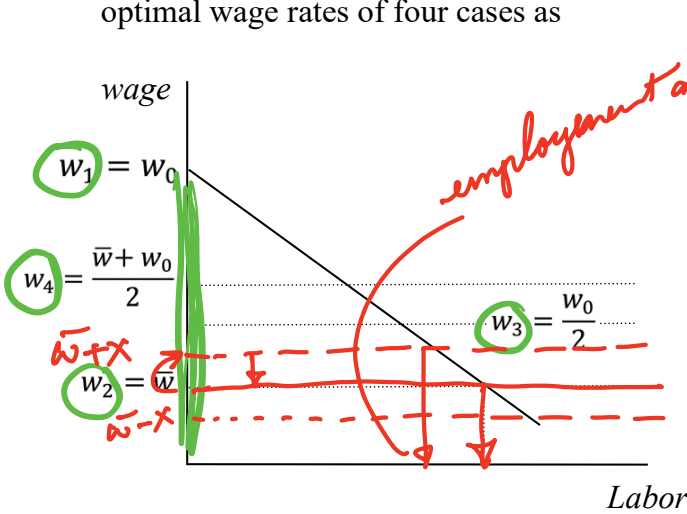


Figure 3.3 Optimal wages for 4 different objectives

Union.  
HW Maximize contribution from the union workers to the union if the contribution is

- a)  $\$X$  per worker — what wage to set
- b)  $x\%$  of the wage earned by each worker  $x\% = 2\%$

HW: p. 61 #2.1, 2.2, 2.3

### 3.2 Profit Maximization in Perfect Competition

A competitive firm with production function

$$Q = f(L, K) = 2L^{0.5}K^{0.5}.$$

Total fees to union =  $F = x \cdot (w \cdot L)$   
 $\Leftrightarrow$  same as in #3       $x$  is a constant.  $\bar{w} + X - X = \bar{w}$   
 so each work will receive after paying the union.

dues payment to union.  
 max its own benefit.  
 number of workers being empty

e)  $X \cdot L$   
 $\max X \cdot L$ ,  $X$  is a constant  
 $\Leftrightarrow \max L$   
 — set the wage to  $\bar{w}$   
 actually the wage to set is  $\bar{w} + X$ .

Assume short-run production with  $K = K_0$ . The profit now has only  $L$  as decision variable.

$$\begin{aligned} \max \pi(L) &= pQ - wL - rK_0 \\ &= p2L^{0.5}K_0^{0.5} - wL - rK_0. \end{aligned}$$

*a constant = fixed cost.*

By first-order sufficient condition,

$$\pi'(L^*) = pL^{*0.5}K_0^{0.5} - w = 0,$$

we have the critical point

$$L^* = \frac{p^2 K_0}{w^2}$$

*demand for Labor.*

independent of  $r$  This can be considered the short-run demand for labor.

This critical point is the maximum point because it satisfies the second-order sufficient condition

$$\pi''(L^*) = -0.5pL^{*-1.5}K_0^{0.5} < 0.$$

The optimal output is  $Q^* = 2L^{*0.5}K_0^{0.5} = \frac{2pK_0}{w}$ , which is a strict global maximum point.

*$\eta_D = \frac{d \ln Q}{d \ln P}$   
here  $Q \rightarrow L$   
 $P \rightarrow w$ .*

**HW** Compute the demand elasticity of the short run demand for labor  $L^* = \frac{p^2 K_0}{w^2}$ .

*$\ln L^* = \ln(P^2 K_0) - 2 \ln w$*

**HW** Redo this example with  $Q = AL^\alpha K_0^\beta$ . Is there any condition on the values of  $\alpha$  and/or  $\beta$ ?

*$\alpha > 0, \beta > 0$ .*

**HW** Baldani, p. 62, #2.4.

### 3.3 Profit Maximization of a Monopoly

A monopoly faces the demand function

$$p = a - bQ,$$

and has the total cost function

$$TC = f + cQ.$$

We have,

$$\begin{aligned} \pi(Q) &= TR - TC = (a - bQ)Q - f - cQ \\ \pi'(Q^*) &= MR - MC = a - 2bQ^* - c = 0 \\ \pi''(Q^*) &= -2b < 0 \end{aligned}$$

*$-(c+t)$   
 $\pi'(Q) = a - 2bQ - c$   
 $\pi''(Q) = -2b < 0$  for any  $c$*

$$Q^* = \frac{a-c}{2b}$$

$$p^* = a - bQ^* = \frac{a+c}{2}$$

$$\pi^* = \dots$$

Note the optimal solution is independent of the fixed cost  $f$ .

HW Baldani, p.62, #2.6, 2.7

**3.4 Taxation on Monopoly** Specific tax  $t$  per unit sold is imposed on a monopoly. The tax collected is given by

$$T = tQ.$$

The profit of the monopoly, assuming a linear demand function, is given by

$$\begin{aligned} \pi(Q) &= TR - TC - T = (a - bQ)Q - f - cQ - tQ \\ &= (a - bQ)Q - \underline{(c + t)Q} - f. \end{aligned}$$

Maximizing this profit, the first- and second-order sufficient conditions are given by

$$\begin{aligned} \pi'(Q^*) &= MR - MC - t = a - 2bQ^* - (c + t) = 0 \\ \pi''(Q^*) &= -2b < 0. \end{aligned}$$

The critical point obtained from the first-order sufficient condition

$$Q^* = \frac{a - c - t}{2b}, \quad a - c - t > 0, \quad \checkmark$$

at the price

$$p^* = a - bQ^* = \frac{a + c + t}{2} \quad \checkmark$$

and the maximal profit is

$$\pi(Q^*) = \frac{(a - c - t)^2}{4b} - f. \quad \checkmark$$

We can perform the Comparative Static Analysis (Sensitivity Analysis). That is, how the optimal solution changes when a parameter changes,

$$\frac{dQ^*}{dt} = -\frac{1}{2b} < 0 \quad \checkmark$$

Implicit f.o.t.

*strict local max of the profit.*  
*Is it global max? Yes if  $\pi$  is concave.*  
*Since  $\pi''(Q) < 0$  for any  $Q$ .*  
 *$\pi$  is strictly concave.*  
*So  $Q^*$  is actually strict global max point.*  
 *$\pi(Q^*) > \pi(Q)$  for any  $Q$  in the domain of  $\pi$ .*

*$\pi''(Q) = -2b < 0$  for any  $Q$ . - strictly concave*

*$c \rightarrow c+t$*

$$\frac{dp^*}{dt} = \frac{1}{2}$$

We also see how the objective function values changes when a parameter changes,

$$\frac{d\pi(Q^*; t)}{dt} = -\frac{a - c - t}{2b} < 0$$

*if  $a > c + t$*

We will see in later chapters that the general theory for sensitivity analyses are based on the Implicit Function Theorem and Envelope Theorem.

$t \rightarrow Q^*$  best output

**HW** What is the specific tax rate  $t$  that maximization tax revenue  $T = tQ^* = t \frac{(a-c-t)}{2b}$ ?

**HW** Baldani, p. 62, #2.8. Redo all above when the tax is ad valorem  $T = tpQ$ .

### 3.5 Profit Maximization of Duopoly (Cournot Model)

A market with two sellers face a common demand function

$$P = a - bQ,$$

where  $Q = q_1 + q_2$ , the sum of quantity sold by both firms. The profit functions of the two firms are given respectively by

$$\begin{aligned} \pi_1(q_1) &= TR_1 - TC_1 = aq_1 - bq_1q_2 - bq_1^2 - cq_1 \\ \pi_2(q_2) &= aq_2 - bq_1q_2 - bq_2^2 - cq_2 \end{aligned}$$

*$q_2$  is taken as constant.*  
 *$q_1$  is constant.*

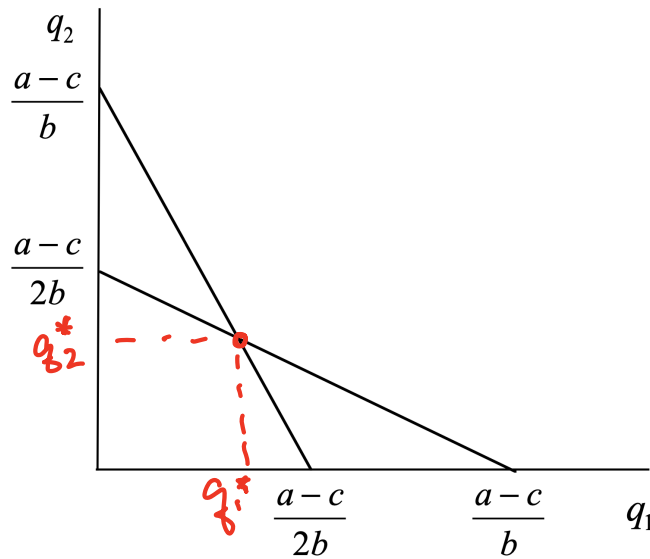
Each firm is assumed to maximize its profit by assuming a given production quantity of the other's. The first-order necessary conditions product the so-called reaction functions as follows.

$$\begin{aligned} \pi'_1(q_1^*) &= a - bq_2 - 2bq_1^* - c = 0 \Rightarrow q_1^* = \frac{a-c}{2b} - \frac{1}{2}q_2 \\ \pi'_2(q_2^*) &= a - bq_1 - 2bq_2^* - c = 0 \Rightarrow q_2^* = \frac{a-c}{2b} - \frac{1}{2}q_1 \end{aligned}$$

Solving the two linear reaction functions, we have the equilibrium

$$q_1^* = \frac{a-c}{3b} = q_2^*$$

This is called a *Nash Equilibrium* because once it is in the equilibrium neither party sees incentive to change.



**Figure 3.4** Reaction functions of two identical firms in a duopoly market.

The market quantity and price are given by

$$Q^* = q_1^* + q_2^* = 2 \left( \frac{a-c}{3b} \right) \checkmark$$

$$P^* = a - bQ^* = \frac{a+2c}{3} \checkmark$$

**HW** Redo the Duopoly model with one of the following changes

- The two firms have different costs  $c_1 \neq c_2$ .
- Each firm has a fixed cost.

**HW** Baldani, p. 63, #2.11.

**3.6 Balanced-Budget Multiplier** Given a simple macroeconomic model as given below, when the government chooses to have a balance budget, the multiplier is equal to 1.

$$Y = C + I + G$$

$$C = a + bY_d, 0 < b < 1$$

$$Y_d = Y - T$$

$$I = I_0 - er, e > 0$$

By substitution, the equilibrium income is

$$Y = a + b(Y - T) + I_0 - er + G$$

$$Y^* = \frac{a - bT + I_0 - er + G}{1 - b}.$$

Taking derivative with respect to the government expenditures and tax, we have

$$\begin{aligned}\frac{\partial Y^*}{\partial G} &= \frac{1}{1 - b} > 0. \\ \frac{\partial Y^*}{\partial T} &= -\frac{b}{1 - b} < 0.\end{aligned}$$

The Balanced Budget Multiplier

$$\frac{\partial Y^*}{\partial G} + \frac{\partial Y^*}{\partial T} = \frac{1}{1 - b} - \frac{b}{1 - b} = 1.$$

We will see a more rigorous treatment in Chapter 5.

**HW** Baldani, p.64, #2.15.