Curve Sketching

1 Curve Sketching

Let f be a relation from X to Y where $X, Y \subseteq \mathbb{R}$.
Recall that we can write f as

$$f = \{(x, y) | x \in X, y \in Y\}$$

or we can write $y = f(x)$, where $x \in X$. We are interested in how to sketch a graph or curve of a give relation or function f.

Definition 1.1 (Graph/Curve)**.** Let X and Y be subsets of $\mathbb{R}$. For a relation $f : X \to Y$, each ordered pair $(x, f(x))$, $x \in X$, can be represented by a point in the Cartesian plane. The collection of all such points is called the **graph or curve of** f.

Curve Sketching
Let f be a relation from X to Y where $X, Y \subseteq \mathbb{R}$. To sketch a curve for f, it is helpful to identify the following characteristics of the curve.

(1) Domain and Range (undefined points)

(2) Intercepts: the x-and y-intercepts

(3) Symmetry: about the x-axis/y-axis/origin

(4) Asymptotes: Vertical/Horizontal/Slant Asymptotes

(5) Increasing/Decreasing Intervals

(6) Local (Relative) Maximum/Minimum

(7) Concavity and Points of Inflection

By using the information obtained from steps (1) to (7)., we can sketch the curve as follows.

- First, draw dashed lines for the asymptotes of the function.
- Then plot the x-and y-intercepts, maximum and minimum points and points of inflection on the graph.
- Sketch the curve between the points, using the intervals of increase and decrease and intervals of concavity.
- Be sure that the graph behaves correctly when approaching asymptotes.

2 Curve Sketching I

2.1 Domain & Range, Intercepts, Symmetry

Remarks

- The steps (1)-(3) can be done by using some basic algebra knowledge.
- For steps (4)-(7), the basic calculus knowledge would be very helpful to identify these characteristics.

Curve Sketching for $f : X \to Y$, $X, Y \subseteq \mathbb{R}$

(1) **Domain & Range** of f: Find the set of all values x such that f is well-defined and the corresponding y. This will be useful when finding vertical asymptotes and determining critical numbers.

(2) **Intercepts:** Find the x-and y-intercepts for $(x, y) \in f$, if possible.

- To find the x-intercept, set $y = 0$ and solve the equation for x.
- To find the y-intercept, set $x = 0$ and solve the equation for y.

(3) **Symmetry:** The curve is...

- **Symmetric about the x-axis** if $\boxed{(x, y) \in f \text{ implies } (x, -y) \in f.}$
- **Symmetric about the y-axis** if $\boxed{(x, y) \in f \text{ implies } (-x, y) \in f.}$
  When f is a function, denoted by $y = f(x)$, the condition above becomes
  $$\boxed{f(x) = f(-x), \quad \forall x \in X}$$
  and the function f is called *"even."*
- **Symmetric about the origin** if $\boxed{(x, y) \in f \text{ implies } (-x, -y) \in f.}$
  When f is a function, denoted by $y = f(x)$, the condition above becomes
  $$\boxed{f(-x) = -f(x), \quad \forall x \in X}$$
  and the function f is called *"odd."*

Example 2.1. Determine the symmetry of the graph for each of the following relations.

1. $f_1 = \{(x, y) | y = x^2\}$

2. $f_2 = \{(x, y) | x = y^2\}$

3. $f_3 = \{(x, y) | y = -x\}$

4. $f_4 = \{(x, y) | x^2 + y^2 = 4\}$

3 Curve Sketching II

3.1 Asymptotes

(4) Asymptotes of a curve can be determined in many different ways. The followings are common two approaches: (i) re-arranging the term and (ii) using the limit.

(i) Re-arranging the term: Consider the relation of the form

$$F(x,y) = C$$

such that $F(x,y))$ can be factored into linear terms, *e.g.*$x-a, y-b, mx+ny$, and C is a nonzero constant.

- **The vertical asymptote**:line $x = a$
- **The horizontal asymptote**: line $y = b$
- **The slant asymptote**: line $mx + ny = 0$

E.g.

- To obtain a **vertical asymptote** (e.g. line $x = a$), arrange the equation in the form $y = f(x)$ and find a such that the *denominator* of $f(a)$ is *zero*.
  Note: In this case, a is not in the domain of f.

- To obtain a **horizontal asymptote** (e.g. line $y = b$), arrange the equation in the form $x = g(y)$ and find b such that the *denominator* of $g(b)$ is *zero*.
  Note: In this case, b is not in the range of f.

(ii) Using the limit: This method is often used when the relation f is a function $f : X \to Y$, $y = f(x), x \in X$.

- **Vertical asymptote**: By using the equation in the form $y = f(x)$,
  $\boxed{x = a}$ is a vertical asymptote if

$$\boxed{\begin{array}{l} \lim_{x\to a} f(x) = \infty \quad \text{or} \quad \lim_{x\to a^+} f(x) = \infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = \infty \quad \text{or} \\ \lim_{x\to a} f(x) = -\infty \quad \text{or} \quad \lim_{x\to a^+} f(x) = -\infty \quad \text{or} \quad \lim_{x\to a^-} f(x) = -\infty. \end{array}}$$

  Alternatively, if it is possible to re-arrange $x = g(y)$, $x = a$ is a vertical asymptote when $\lim_{y\to\infty} g(y) = a \quad \text{or} \quad \lim_{y\to-\infty} g(y) = a$.

- **Horizontal asymptote**: By using the equation in the form $y = f(x)$,
  $\boxed{y = b}$ is a horizontal asymptote if

$$\boxed{\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.}$$

  Alternatively, if it is possible to re-arrange $x = g(y)$, $y = b$ is a horizontal asymptote when
  $\lim_{y\to b} g(y) = \infty \quad \text{or} \quad \lim_{y\to b^+} g(y) = \infty \quad \text{or} \quad \lim_{y\to b^-} g(y) = \infty \quad \text{or}$
  $\lim_{y\to b} g(y) = -\infty \quad \text{or} \quad \lim_{y\to b^+} g(y) = -\infty \quad \text{or} \quad \lim_{y\to b^-} g(y) = -\infty.$

- **Slant asymptote**: By using the equation in the form $y = f(x)$, the line $\boxed{y = mx + b}$ is a slant asymptote if

$$\boxed{\lim_{x\to\infty}[f(x) - (mx + b)] = 0 \quad \text{or} \quad \lim_{x\to-\infty}[f(x) - (mx + b)] = 0.}$$

Slant asymptotes for rational functions:

Example 3.1. Consider the relation defined by

$$f = \{(x, y)|x^2 + xy - x - y - 3 = 0\}.$$

Find all asymptotes for the curve of f.

Example 3.2. Consider a function

$$f(x) = \frac{x^2 + 5x + 4}{x^2}.$$

1. Find domain and range of f.
2. Find x-intercepts and y-intercepts.
3. Determine the symmetry of f.
4. Find the asymptotes for f (if any).

The characteristics (5)-(7), increasing/decreasing, extrema, and concavity, of the curves of a relation $f : X \to Y$ can generally be identified when f is a *continuous* and *differentiable* function on a subset of its domain $x \subseteq \mathbb{R}$ by using some basic *calculus*. The followings are some calculus required to used for identifying these characteristics.

Definition 3.1. A critical number of a function $f : X \to Y$ is a number c in its domain X for which

$$f'(c) = 0, \qquad \text{or} \qquad f'(c) \quad \text{does not exist.}$$

Example 3.3. Find the domain and the critical numbers for each of the following functions.

- $f(x) = (x+4)^{2/3}$
- $f(x) = \frac{x^2}{x-1}$.

3.2 Increasing/Decreasing

(5) **Increasing/Decreasing Intervals:** (defined below) can be identified by using the first derivative from the following theorem.

Definition 3.2 (Increasing/Decreasing Function)**.** Let $f : X \to Y$ be a function $y = f(x)$, $X, Y \subseteq \mathbb{R}$. Let $S \subseteq X$.

- f is increasing on S if and only if, for all $x_1, x_2 \in S$, if $x_1 < x_2$, then $f(x_1) < f(x_2)$.
- f is decreasing on S if and only if, for all $x_1, x_2 \in S$, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.

Theorem 3.1. Test for Increasing/Decreasing Let f be a function that is continuous on $[a, b]$ and differentiable on (a, b).

- If $f'(x) > 0$ for all x in (a, b), then f is increasing on $[a, b]$.
- If $f'(x) < 0$ for all x in (a, b), then f is decreasing on $[a, b]$.
- If $f'(x) = 0$ for all x in $[a, b]$, then f is a constant on the interval.

3.3 Extrema

(6) **Extrema:** (defined below) of f can be identified by using the first derivatives and the second derivatives from the following theorems.

Let $f : X \to Y$, $X, Y \subset \mathbb{R}$ be a function.

Definition 3.3 (Relative Extrema: Maximum/Minimum)**.**
- A number $f(c)$ is a **relative maximum** of a function f if $f(x) \leq f(c)$ for every x in *some* open interval that contains c.
- A number $f(c)$ is a **relative minimum** of a function f if $f(x) \geq f(c)$ for every x in *some* open interval that contains c.

Theorem 3.2. Relative Extrema Occur at Critical numbers If a function f has a relative extremum at $x = c$, then c is a critical number.

From the previous theorem, we can find an extremum by first find all critical numbers c of f first. Then, we can use two following approaches to identify if each critical number gives maximum, minimum, or it does not give any extremum.

- The First Derivative Test for Relative Extrema
- The Second Derivative Test for Relative Extrema

Theorem 3.3. First Derivative Test for Relative Extrema Let f be continuous on $[a, b]$ and differentiable on (a, b) except possibly at the critical number c.

(i) If $f'(x)$ changes from positive to negative at c, then $f(c)$ is a relative **maximum**.

(ii) If $f'(x)$ changes from negative to positive at c, then $f(c)$ is a relative **minimum**.

(iii) If $f'(x)$ has the same algebraic sign on each side of c, then $f(c)$ is **not an extremum**.

Theorem 3.4. Theorem: Second Derivative Test for Relative Extrema Let f be a function for which the second derivative f'' exists on an interval (a, b) that contains the critical number c.

(i) If $f''(c) > 0$, then $f(c)$ is a relative **minimum**.

(ii) If $f''(c) < 0$, then $f(c)$ is a relative **maximum**.

(iii) If $f''(c) = 0$, the test fails and $f(c)$ may or may not be a relative extremum. In this case, use the First derivative Test.

Notice that, to find an extremum, we have to first find all critical numbers c of f.

Example 3.4. Consider the function $f(x) = x^3 - 3x^2 - 9x + 2$

1. Find domain of f.

2. Find x-intercepts and y-intercepts (if any).

3. Determine the symmetry of f. Determine whether f is even or odd.

4. Find the asymptotes for f (if any).

5. Find the critical number of f.

6. Determine the intervals on which f is increasing and decreasing.

7. Determine the extrema (maximum and minimum) of f by using
   - the First Derivative Test,
   - the Second Derivative test.

3.4 Concavity & Inflection Point

(7) **Concavity and the point of inflection:** (defined below) of f can be identified by using the the second derivatives from the following theorem.

Let $f : X \to Y$ be a function with $X, Y \subseteq \mathbb{R}$.

Definition 3.4 (Concavity: concave up/concave down)**.** Let f be a differentiable function on an interval (a, b).

(i) If f' is an increasing function on (a, b), then the graph of f is **concave up** on the interval.

(ii) If f' is an decreasing function on (a, b), then the graph of f is **concave down** on the interval.

Definition 3.5 (Point of Inflection)**.** Let f be a continuous on an interval (a, b) containing the number c. A point $(c, f(c))$ is a **point of inflection** of the graph f if there is a tangent line at $(c, f(c))$ and the graph changes concavity at this point.

Theorem 3.5. Test for Concavity Let f be function for which f'' exists on (a, b).

(i) If $f''(x) > 0$ for all x in (a, b), then the graph of f is concave up on (a, b).

(ii) If $f''(x) < 0$ for all x in (a, b), then the graph of f is concave down on (a, b).

Theorem 3.6. Point of Inflection
If $(c, f(c))$ is a point of inflection for the graph of a function f, then either

$$f''(c) = 0 \qquad \text{or} \qquad f''(c) \text{ does not exist.}$$

Example 3.5. Consider the function $f(x) = 4x^4 - 4x^2$.

1. Find domain of f.
2. Find x-intercepts and y-intercepts (if any).
3. Determine the symmetry of f. Determine whether f is even or odd.
4. Find the asymptotes for f (if any).
5. Find the critical numbers of f.
6. Determine the intervals on which f is increasing and decreasing.
7. Determine the relative extrema (maximum and minimum) of f
8. Use the information above to sketch the graph of f.

Example (continued)