Note Title 8/24/2016

Last time.

2.1 First-order Linear DE

2.1.1 constant Coefficient and constant term

$$y'(t) + a\,y(t) = b \qquad -(4)$$

$$y(t) = y_c + y_p$$

✓✓ $$\boxed{y(t) = Ae^{-at} + \frac{b}{a}} \qquad a \neq 0$$

$$\boxed{y(t) = \left[y(0) - \frac{b}{a}\right]e^{-at} + \frac{b}{a} \qquad a \neq 0}$$

$a = 0$

we set $y_p = k \rightarrow y'(t) = 0$

sub into (4) $\quad 0 + 0(k) = b$

$0 = b$

not verified

$y_p = k.t \rightarrow y'(t) = k$

sub. into (4) $\quad k + (0)(kt) = b$

$k = b$

$\therefore\ y_p = bt$

$$y(t) = Ae^{-at} + bt$$

$$y(t) = Ae^{0} + bt$$

Gen. Sol^{n} $\boxed{y(t) = A + bt}$ $a = 0$

Def. Sol^{n} $\boxed{y(t) = y(0) + bt}$ $a = 0$

2.1.2. Dynamics of Mkt Price

$$P^* = \frac{\alpha + \gamma}{\beta + \delta} \quad - (10)$$

$$\frac{dP}{dt} = j(Q_d - Q_s) \quad ; \; j > 0 \quad - (11)$$

$$\frac{dP}{dt} = 0 \iff \boxed{Q_d = Q_s} \quad P^*$$

Equilibrium Price (P^*)

1) The mkt-clearing price ⇒ Price that equates Q_d and Q_s

2) The intertemporal sense ; P^* is constant overtime

sub Q_d, Q_s ⇒ into (11)

$$\frac{dP}{dt} = j\left[\overbrace{(\alpha - \beta P)}^{Q_d} - \overbrace{(-\gamma + \delta P)}^{Q_s}\right]$$

$$\frac{dP}{dt} = j(\alpha + \gamma) - j(\beta + \delta)P$$

$$\frac{dP}{dt} + j(\beta + \delta)P = j(\alpha + \gamma)$$

$$\frac{dy}{dt} + a\,y(t) = b \Rightarrow$$

$$P(t) = \left[P(0) - \frac{j(\alpha+\gamma)}{j(\beta+\delta)}\right] e^{-j(\beta+\delta)t} + \frac{j(\alpha+\gamma)}{j(\beta+\delta)}$$

$$P(t) = \left[P(0) - \frac{(\alpha+\gamma)}{(\beta+\delta)}\right] e^{-j(\beta+\delta)t} + \frac{(\alpha+\gamma)}{(\beta+\delta)}$$

$$P(t) = \left[P(0) - P^*\right] e^{-kt} + P^* \; ; \; k = j(\beta+\delta) \qquad \text{— (11)'}$$

$k = j(\beta+\delta) > 0$

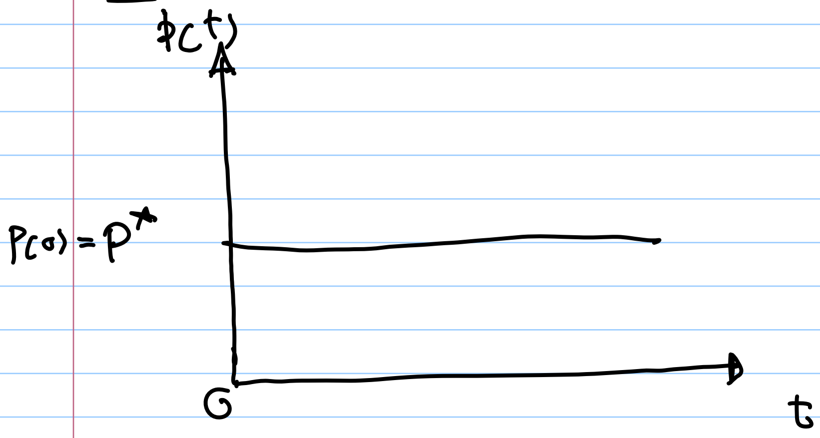
$t \to \infty \Rightarrow e^{-kt} \to 0$

$\Rightarrow P(t) \to P^*$

$P(t)$ converges to the equilibrium level P^*

P^* is Dynamically stable

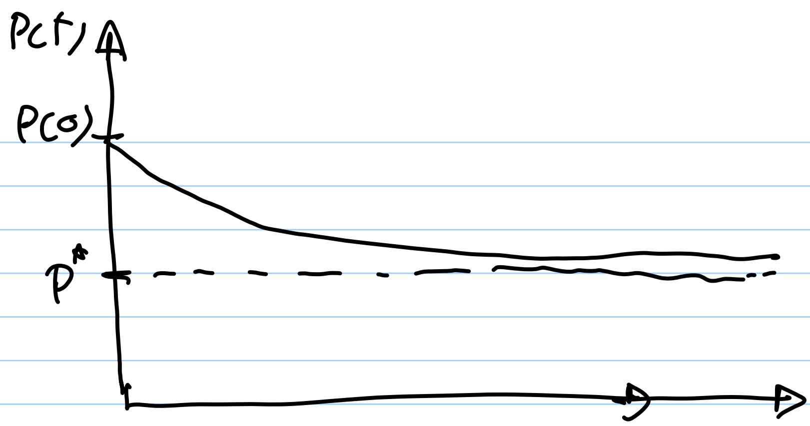
Case 1 : $P(0) = P^*$



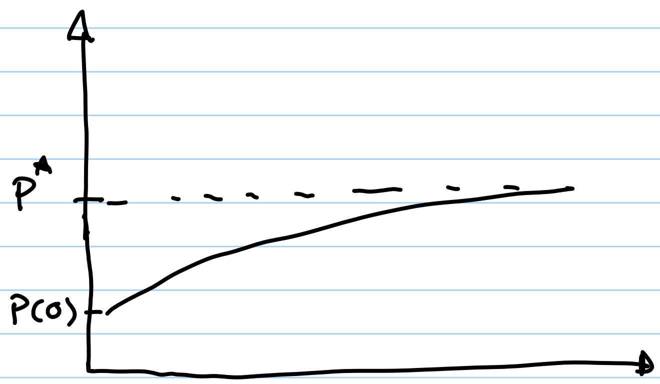
$$P(t) = [P(0) - P^*]e^{-bt} + P^*$$

$$P(t) = P^*$$

Case 2 $P(0) > P^*$



Case 3 $P(0) < P^*$



$$y(t) = y_c + y_p$$

$$= \left[y(0) - \frac{b}{a}\right]e^{-at} + \frac{b}{a}$$

y_p = the intertemporal equilibrium of $y(t)$

y_c = the deviation from equilibrium

2.1.3 Variable coefficient and variable term

$$\frac{dy}{dt} + u(t).\, y(t) = w(t) \qquad -(12)$$

$u(t)$, $w(t)$ = continuous f^n of t

⊙ Homogeneous case

$$w(t) = 0$$

$$\frac{dy(t)}{dt} + u(t).\, y(t) = 0 \qquad -(13)$$

$$\frac{1}{y}\frac{dy}{dt} = -u(t)$$

$$\int \frac{1}{y}\frac{dy}{dt}\, dt = \int -u(t)\, dt$$

$$\ln|y| + C = -\int u(t)\, dt$$

$$\ln|y| = -C - \int u(t)\, dt$$

$$e^{\ln|y|} = e^{-c} e^{-\int u(t)dt}$$

$$y(t) = A e^{-\int u(t)dt} \quad ; A = e^{-c} \qquad \text{— (14)}$$

(14) is the general sol^{n} of (13)

Ex

$$y'(t) + 3t^2 y(t) = 0$$

$$u(t) = 3t^2$$

$$\int u(t)\,dt = \int 3t^2\,dt$$

$$= t^3 + C$$

$$y(t) = A e^{-\int u(t)\,dt}$$

$$= A e^{-(t^3+c)}$$

$$= A e^{-c} e^{-t^3}$$

$$\boxed{y(t) = B e^{-t^3}} \quad ; \; B = A e^{-c}$$

Check

$$y'(t) = -3t^2 B e^{-t^3}$$

sub into the original DE

$$-3t^2 B e^{-t^3} + (3t^2)(B e^{-t^3}) = 0$$

$$\underbrace{\qquad}_{y'(t)} \qquad \underbrace{\qquad}_{y(t)}$$

$$0 \quad = \quad 0$$

@ The Nonhomogeneous Case

$$w(t) \neq 0$$

$$y(t) = e^{-\int u(t)\,dt}\left[A + \int w(t)\, e^{\int u(t)\,dt}\,dt\right] \qquad -(15)$$

To get (15), we use the concept of EXACT DE

2.1.4 Exact Differential Equations

consider

$$g(y,t)\cdot\frac{dy}{dt} + f(y,t) = 0 \qquad -(16)$$

suppose we have a f^n $F(y,t)$ such that

$$\left.\begin{aligned} \frac{\partial F(y,t)}{\partial t} &= f(y,t) \\ \frac{\partial F(y,t)}{\partial y} &= g(y,t) \end{aligned}\right\} \qquad -(17)$$

This implies the total derivatives of $F(y,t)$

$$dF(y,t) = \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial t} dt$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial F}{\partial t} \cdot \frac{dt}{dt}$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial y} \frac{dy}{dt} + \frac{\partial F}{\partial t}$$

(17)

$$\frac{dF}{dt} = \underbrace{g(y,t) \frac{dy}{dt} + f(y,t)}_{\text{LHS of (16)}}$$

$$\frac{dF(y,t)}{dt} = 0 \Leftrightarrow F(y,t) = \text{constant } C$$

The general sol^n should be $\boxed{F(y,t) = C}$

Ex

$$2ty\,dy + y^2 dt = 0 \quad ✓✓✓$$

or $$2ty \frac{dy}{dt} + y^2 = 0$$

$$g(y,t) \cdot \frac{dy}{dt} + f(y,t) = 0$$

$$g(y,t) = 2ty = \frac{\partial F}{\partial y} \rightarrow \frac{\partial^2 F}{\partial y \partial t} = 2y = \frac{\partial g}{\partial t} - (a)$$

$$f(y,t) = y^2 = \frac{\partial F}{\partial t} \longrightarrow \frac{\partial^2 F}{\partial t \partial y} = 2y = \frac{\partial f}{\partial y} \quad -(b)$$

$\boxed{\text{we are going to solve for } y(t)}$.

ψ - psi

ϕ - phi

ω

Let $g(y,t) = \frac{\partial F}{\partial y} = M$

$$F(y,t) = \int M\,dy + \psi(t)$$

$$C = \int M\,dy + \psi(t)$$

step i

$$F(y,t) = \int M\,dy + \psi(t)$$

$$= \int 2ty\,dy + \psi(t)$$

$$F(y,t) = y^2 t + C_1 + \psi(t) \quad -(*)$$

step ii

derivative $(*)$

w.r.t. t

$$\frac{\partial F(y,t)}{\partial t} = y^2 + \psi'(t) \quad -(**)$$

From (b)

$$f(y,t) = y^2 = \frac{\partial F}{\partial t} \quad -(b)$$

$$y^2 = y^2 + \psi'(t)$$

$$\psi'(t) = 0$$

step iii $\quad \psi(t) = \int \psi'(t)\,dt$

$$= \int 0\,dt$$

$$\psi(t) = C_2$$

step iv $\quad$ go back to step i and sub $\psi(t) = C_2$ into $(*)$

$$F(y,t) = y^2 t + C_2 + C_1$$

$$F(y,t) = y^2 t + C_3 \quad ; \quad C_3 = C_2 + C_1$$

we know $F(y,t) = C$

$$C = y^2 t + C_3$$

$$y^2 t = C_4 \quad ; \quad C_4 = C - C_3$$

$$y^2 = C_4 t^{-1}$$

$$\therefore \boxed{y(t) = C_5 t^{-\frac{1}{2}}} \quad ; \quad C_5 = (C_4)^{\frac{1}{2}}$$

check

$$2ty\frac{dy}{dt} + y^2 = 0 \qquad y'(t) = -\frac{1}{2} C_5 t^{-\frac{3}{2}}$$

$$2t\left(c_5 t^{-\frac{1}{2}}\right)\left(-\frac{1}{2} c_5 t^{-\frac{3}{2}}\right) + \left(-\frac{1}{2} c_5 t^{-\frac{3}{2}}\right)^2 = 0$$

$$\frac{dy}{dt} + u(t)\, y(t) = w(t)$$

$$\frac{dy}{dt} + (uy - w) = 0 \quad \text{—} \quad (\star)$$

From

$$g(y,t)\,\frac{dy}{dt} + f(y,t) = 0$$

not exact

$$g(y,t) = 1 = \frac{\partial F}{\partial y} \rightarrow \frac{\partial^2 F}{\partial y \partial t} = 0$$

$$f(y,t) = u(t)\,y(t) - w(t) = \frac{\partial F}{\partial t} \rightarrow \frac{\partial^2 F}{\partial t \partial y} = u(t)$$

We can make $(\star)$ exact DE by premultiplying by $I(t)$ "integrating factor"

$(\star)$ becomes

$$\underbrace{I(t)}_{g(y,t)}\frac{dy}{dt} + \underbrace{I(t)\left[u(t)y(t) - w(t)\right]}_{f(y,t)} = 0$$

$$\frac{\partial^2 F}{\partial y \partial t} = \frac{\partial g}{\partial t} = \frac{dI(t)}{dt}$$

$$\frac{\partial^2 F}{\partial t \partial y} = \frac{\partial f}{\partial y} = I(t)\,u(t)$$

$$\frac{dI}{dt} = I(t)\,u(t)$$

$$\frac{1}{I}\frac{dI}{dt} = u(t)$$

$$\int \frac{1}{I}\frac{dI}{dt}\,dt = \int u(t)\,dt$$

$$\ln|I| = -c + \int u(t)\,dt$$

$$e^{\ln|I|} = e^{-c}\,e^{\int u(t)\,dt}$$

$$\boxed{I(t) = A\,e^{\int u(t)\,dt}} \qquad A = e^{-c}$$

Assuming that $A = 1$

(A) becomes

$$\overbrace{e^{\int u(t)\,dt}}^{I(t)}\frac{dy}{dt} + \overbrace{e^{\int u(t)\,dt}}^{I(t)}\left[u(t)\,y(t) - w(t)\right] = 0$$

$$g(y,t) = \frac{\partial F}{\partial y} = M = e^{\int u(t)\,dt}$$

$$f(y,t) = \frac{\partial F}{\partial t} = e^{\int u(t)\,dt}\left[u(t)\,y(t) - w(t)\right]$$

step i $\quad F(y,t) = \int M\,dy + \psi(t) \qquad (A1)$

$$= \int e^{\int u(t)\,dt}\,dy + \psi(t)$$

$$= \left[e^{\int u(t)\,dt} \cdot y + C \right] + \psi(t)$$