Quiz 5

1. Consider the functions f and g defined as

$$f(x) = \begin{cases} \frac{\sqrt{x^2-1}}{\sqrt{2+x}}, & -2 < x < -1 \\ 0, & x \geq -1. \end{cases} \qquad \text{and} \qquad g(x) = \frac{1}{|x+1|}.$$

(a) Determine whether or not $R_f \subseteq D_g$. Briefly explain your answer.
(b) Determine whether or not $R_g \subseteq D_f$. Briefly explain your answer.
(c) Determine $g \circ f$ and $f \circ g$ (if possible).

Solution:
To find the domain of $f(x)$, we notice that when $-2 < x < -1$, we have $x^2 - 1 > 0$ and $2 + x > 0$ and $\frac{\sqrt{x^2-1}}{\sqrt{2+x}}$ is well-defined for $-2 < x < -1$. That is, the domain of f is $\boxed{D_f = (-2,-1) \cup [-1,\infty) = (-2,\infty)}$
To find the domain of $g(x)$, we only need to have $|x+1| \neq 0$ or $x \neq -1$. That is, the domain of g is $\boxed{D_g = \mathbb{R} - \{-1\}}$.

(a) For $x \in (-2,-1)$, $f(x) = \frac{\sqrt{x^2-1}}{\sqrt{2+x}} > 0$ and for $x \in (-1,\infty)$, $f(x) = 0$.
So, for $x \in D_f$, $f(x) \in R_f \subseteq [0,\infty)$. That is,

$$R_f \subseteq [0,\infty) \subseteq \mathbb{R} - \{-1\} = D_g \quad \Rightarrow \quad R_f \subseteq D_g.$$

■

(b) For $x \in D_g = \mathbb{R} - \{-1\}$, we have $|x+1| > 0$ and

$$g(x) = \frac{1}{|x+1|} > 0 \quad \Rightarrow \quad g(x) \in R_g \subseteq (0,\infty) \subset (-2,\infty) = D_f \quad \Rightarrow \quad R_g \subseteq D_f$$

■

(c)
- Since $R_f \subseteq D_g$ from (a), then $D_{g \circ f} = D_f = (-2,\infty)$ and it is possible to find $g \circ f$ as follows.
For $-2 < x < -1$, $(g \circ f)(x) = g(f(x)) = g(\frac{\sqrt{x^2-1}}{\sqrt{2+x}}) = 1/|\frac{\sqrt{x^2-1}}{\sqrt{2+x}} + 1|$.

For $x \geq -1$, $(g \circ f)(x) = g(f(x)) = g(0) = \frac{1}{|0+1|} = 1$. Hence,

$$(g \circ f)(x) = g(f(x)) = \begin{cases} \frac{1}{|\frac{\sqrt{x^2-1}}{\sqrt{2+x}}+1|}, & -2 < x < -1 \\ 1, & x \geq -1. \end{cases}$$

■

-Since $R_g \subseteq D_f$ from (a), then $D_{f \circ g} = D_g = \mathbb{R} - \{-1\}$ and it is possible to find $f \circ g$ as follows.
Notice that, since $g(x) = \frac{1}{|x+1|} > 0$, then $g(x) > -1$ for all $x \in D_{f \circ g}$ and we must use the second formula of f:

$$(f \circ g)(x) = f(g(x)) = f(\frac{1}{|x+1|}) = 0 \qquad \forall x \in D_{f \circ g} = \mathbb{R} - \{-1\}.$$

■