2. Assume there is an economy with k states of nature and where the following asset pricing formula holds:

$$\begin{aligned} P_a &= \sum_{s=1}^{k} \pi_s m_s X_{sa} \\ &= E\left[m X_a\right] \end{aligned}$$

Let an individual in this economy have the utility function $\ln(C_0) + E\left[\delta \ln(C_1)\right]$, and let C_0^* be her equilibrium consumption at date 0 and C_s^* be her equilibrium consumption at date 1 in state s, $s = 1, ..., k$. Denote the date 0 price of elementary security s as p_s, and derive an expression for it in terms of the individual's equilibrium consumption.

From definition of elementary securities, $p_s = \pi_s m_s (1)$

$m_s = \dfrac{\delta U'(C_s^*)}{U'(C_0^*)} = \delta \dfrac{C_0^*}{C_s^*}$. we get that

$p_s = \delta \pi_s \dfrac{C_0^*}{C_s^*}$

3. Consider the one-period consumption-portfolio choice problem. The individual's first-order conditions lead to the general relationship

$$1 = E\left[m_{01} R_s\right]$$

where m_{01} is the stochastic discount factor between dates 0 and 1, and R_s is the one-period stochastic return on any security in which the individual can invest. Let there be a finite number of date 1 states where π_s is the probability of state s. Also assume markets are complete and consider the above relationship for primitive security s; that is, let R_s be the rate of return on primitive (or elementary) security s. The individual's elasticity of intertemporal substitution is defined as

$$\varepsilon^I \equiv \frac{R_s}{C_s/C_0} \frac{d\left(C_s/C_0\right)}{dR_s}$$

where C_0 is the individual's consumption at date 0 and C_s is the individual's consumption at date 1 in state s. If the individual's expected utility is given by

$$U\left(C_0\right) + \delta E\left[U\left(\tilde{C}_1\right)\right]$$

where utility displays constant relative risk aversion, $U\left(C\right) = C^\gamma/\gamma$, solve for the elasticity of intertemporal substitution, ε^I.

$$m_{01} = \frac{\delta U'(C_1)}{U'(C_0)} = \delta\left(\frac{C_1}{C_0}\right)^{\gamma-1}.$$

For primitive security s, FOC is

$$1 = \pi_s \delta \left(\frac{C_s}{C_0}\right)^{\gamma-1} R_s$$

Total differentiate :

$$0 = \pi_s \delta (\gamma-1) \left(\frac{C_s}{C_0}\right)^{\gamma-2} R_s \, \partial\left(\frac{C_s}{C_0}\right) + \pi_s \left(\frac{C_s}{C_0}\right)^{\gamma-1} \partial R_s$$

$$\frac{R_s}{\left(\frac{C_s}{C_0}\right)} \cdot \frac{\partial\left(\frac{C_s}{C_0}\right)}{\partial R_s} = \frac{1}{1-\gamma}$$

4. Consider an economy with $k = 2$ states of nature, a "good" state and a "bad" state.[16] There are two assets, a risk-free asset with $R_f = 1.05$ and a second risky asset that pays cashflows

$$X_2 = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

The current price of the risky asset is 6.

a. Solve for the prices of the elementary securities p_1 and p_2 and the risk-neutral probabilities of the two states.

b. Suppose that the physical probabilities of the two states are $\pi_1 = \pi_2 = 0.5$. What is the stochastic discount factor for the two states?

a. $P = \begin{bmatrix} 1/1.05 \\ 6 \end{bmatrix}$

$X = \begin{bmatrix} 1 & 10 \\ 1 & 5 \end{bmatrix}$

$[p_1 \;\; p_2] = P' X^{-1} = \begin{bmatrix} \frac{1}{1.05} & 6 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0.2 & -0.2 \end{bmatrix}$

$= [0.2476 \;\; 0.7048]$

$\therefore$ Risk neutral prob. are : $p_1 R_f = 0.26$

$p_2 R_f = 0.74$

b. $m_1 = p_1/\pi_1 = 0.495$

$m_2 = p_2/\pi_2 = 1.410$

6. This question asks you to relate the stochastic discount factor pricing relationship to the CAPM. The CAPM can be expressed as

$$E[R_i] = R_f + \beta_i \gamma$$

where $E[\cdot]$ is the expectation operator, R_i is the realized return on asset i, R_f is the risk-free return, β_i is asset i's beta, and γ is a positive market risk premium. Now, consider a stochastic discount factor of the form

$$m = a + bR_m$$

where a and b are constants and R_m is the realized return on the market portfolio. Also, denote the variance of the return on the market portfolio as σ_m^2.

a. Derive an expression for γ as a function of a, b, $E[R_m]$, and σ_m^2. (Hint: you may want to start from the equilibrium expression $0 = E[m(R_i - R_f)]$.)

b. Note that the equation $1 = E[mR_i]$ holds for all assets. Consider the case of the risk-free asset and the case of the market portfolio, and solve for a and b as a function of R_f, $E[R_m]$, and σ_m^2.

c. Using the formula for a and b in part (b), show that $\gamma = E[R_m] - R_f$.

a.

$$\begin{aligned}
0 &= E(m(R_i - R_f)) \\
&= E[(a + bR_M)(R_i - R_f)] \\
&= aE(R_i) - aR_f + bE(R_M R_i) - bR_f E(R_M) \\
&= a(E(R_i) - R_f) + b(E(R_M)E(R_i) + cov(R_M, R_i) - R_f E(R_M)) \\
&= (E(R_i) - R_f)(a + bE(R_M)) + b\,cov(R_M, R_i)
\end{aligned}$$

$$\therefore \quad E(R_i) - R_f = \frac{-b\,cov(R_M, R_i)}{a + bE(R_M)}$$

$$= -\frac{cov(R_M, R_i)}{\sigma_M^2} \cdot \frac{b\sigma_M^2}{a + bE(R_M)}$$

$$= -\beta_i \frac{b\sigma_M^2}{a + bE(R_M)}$$

$$\therefore \quad \gamma = \frac{-b\sigma_M^2}{a + bE(R_M)}$$

b. For risk-free asset, $\frac{1}{R_f} = E(a + bR_M)$

$$a = \frac{1}{R_f} - bE(R_M) \quad -①$$

For market portfolio, $1 = E[(a + bR_M)R_M]$

$$= aE[R_M] + bE[R_M^2]$$

$$= aE(R_M) + b(\sigma_M^2 + E(R_M)^2)$$

Sub. ①: $1 = \left(\frac{1}{R_f} - bE(R_M)\right)E(R_M) + b(\sigma_M^2 + E(R_M)^2)$

$$1 = \frac{E(R_M)}{R_f} + b\sigma_M^2$$

$$b = -\frac{E(R_M) - R_f}{R_f\sigma_M^2}$$

c. $a + bE(R_m) = \dfrac{\sigma_M^2 + E(R_M)(E(R_M) - R_f) - E(R_M)(E(R_M) - R_f)}{R_f\sigma_M^2}$

$$= \frac{1}{R_f}$$

$$\gamma = \frac{-b\sigma_M^2}{a + bE(R_M)}$$

$$= \frac{E(R_M) - R_f}{R_f \sigma_M^2} \sigma_M^2 R_f$$

$$= E(R_M) - R_f$$