

4 Testing Hypotheses about a Single Linear Combination of the Parameter

Consider

$$\log(\text{wage}) = \beta_0 + \beta_1 jc + \beta_2 univ + \beta_3 \text{exper} + u$$

where *jc* = number of years attending a two-year college
univ = number of years at a four-year college
exper = months in the workforce.

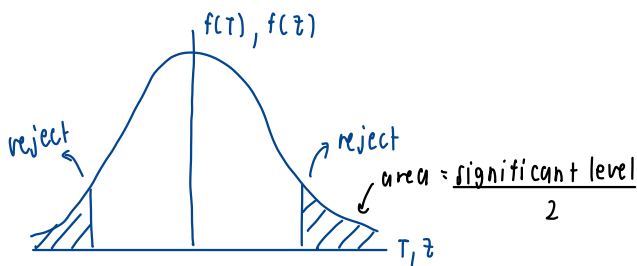
We want to test whether $\beta_1 = \beta_2$. *if the returns from 1 more year of education at a Junior college is the same as that of the university*

$H_0 : \beta_1 = \beta_2 \rightarrow H_0 : \beta_1 - \beta_2 = 0$

against

$H_a : \beta_1 \neq \beta_2 \rightarrow H_a : \beta_1 - \beta_2 \neq 0$

2 tailed test



$$t = \frac{(\hat{\beta}_1 - \hat{\beta}_2) - 0}{\text{s.e.}(\hat{\beta}_1 - \hat{\beta}_2)}$$

we compute this t-statistic and compare with the critical value

where $\text{s.e.}(\hat{\beta}_1 - \hat{\beta}_2) = \sqrt{\text{var}(\hat{\beta}_1 - \hat{\beta}_2)}$
not very straight forward to calculate
 $= \sqrt{\widehat{\text{var}}(\hat{\beta}_1) + \widehat{\text{var}}(\hat{\beta}_2) - 2\widehat{\text{cov}}(\hat{\beta}_1, \hat{\beta}_2)}$
we use a variable transformation trick

4 2' 2' 2' 2'

another possible hypothesis test (one-tailed alternative)

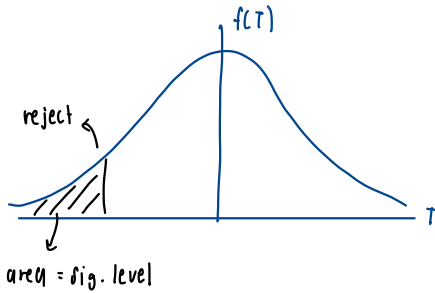
$$H_0 : \beta_1 = \beta_2 \rightarrow H_0 : \beta_1 - \beta_2 = 0$$

$$H_a : \beta_1 < \beta_2 \rightarrow H_a : \beta_1 - \beta_2 < 0$$

It is assumed that β_1 would not be more than β_2 (returns to a 2-year college would never be more than returns to university education)

$$t = \frac{(\hat{\beta}_1 - \hat{\beta}_2) - 0}{\text{s.e.}(\hat{\beta}_1 - \hat{\beta}_2)}$$

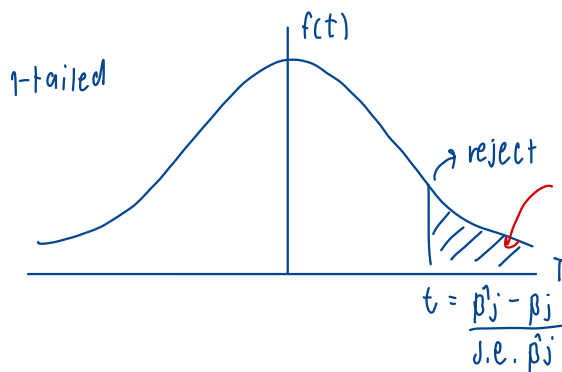
Then, go to the extra note.



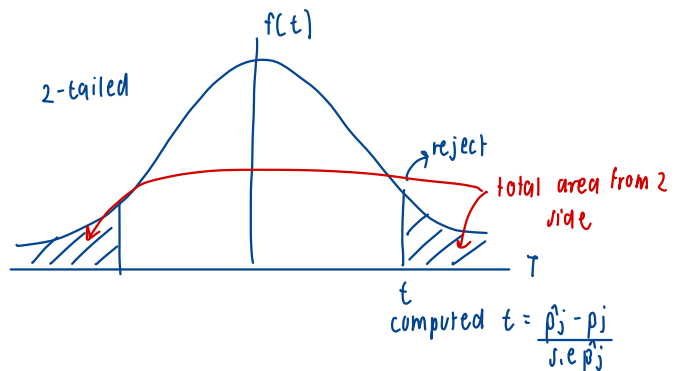
Handwritten red text: $\beta_1 < \beta_2$

5 Computing p-Values for t-Tests

What is the significance level given the computed t-statistics?



This shaded area in the rejection is the p-value



p-value : $P(|T| > |t|)$

T = t-distributed random variable with d.f. = $n - k - 1$

t = computed t-statistic

p-value = probability that a random T value will be greater (in the $| |$ term) than our t in the H_0 test.

consider the multiple regression model, assume MLR 1-6 are satisfied.

বিষয়

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + u$$

You would like to test the $H_0: \beta_1 - 3\beta_2 = 1$

H_a : otherwise is true

step ↘

① write the t-statistic for testing H_0

$$t = \frac{(\hat{\beta}_1 - 3\hat{\beta}_2) - 1}{\text{S.E.}(\hat{\beta}_1 - 3\hat{\beta}_2)}$$

② define $\theta_1 = \beta_1 - 3\beta_2 \rightarrow H_0: \theta_1 = 1$

$H_a: \theta_1 \neq 1$

$$t = \frac{\hat{\theta}_1 - 1}{\text{S.E.}(\hat{\theta}_1)}$$

we need our regression to have θ_1 in it. so, OLS or OLS estimation will automatically give $\hat{\theta}_1$ and S.E. $\hat{\theta}_1$

now, $\hat{\beta}_1 = \hat{\theta}_1 + 3\hat{\beta}_2$

or $\beta_1 = \theta_1 + 3\beta_2$

substitution in the main regression and get

$$Y = \beta_0 + (\theta_1 + 3\beta_2)x_1 + \beta_2 x_2 + \beta_3 x_3 + u$$

$$= \beta_0 + \theta_1 x_1 + 3\beta_2 x_1 + \beta_2 x_2 + \beta_3 x_3 + u$$

$$= \beta_0 + \theta_1 x_1 + \beta_2 (x_2 + 3x_1) + \beta_3 x_3 + u$$

Now, the explanatory variables are going to be x_1 , $x_2 + 3x_1$, and x_3

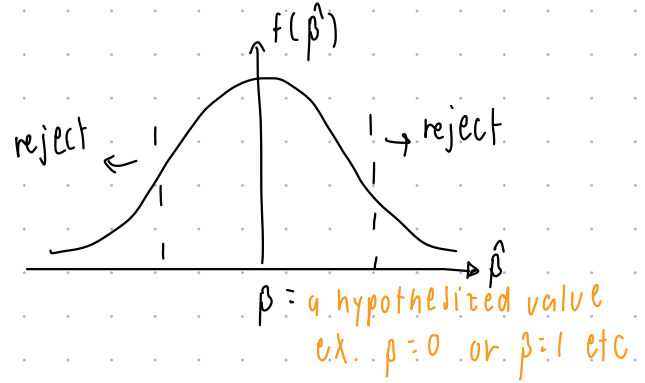
we can calculate $t = \frac{\hat{\theta}_1 - 1}{\text{S.E.}(\hat{\theta}_1)}$

Inference : Hypothesis testing about " β " the true parameter

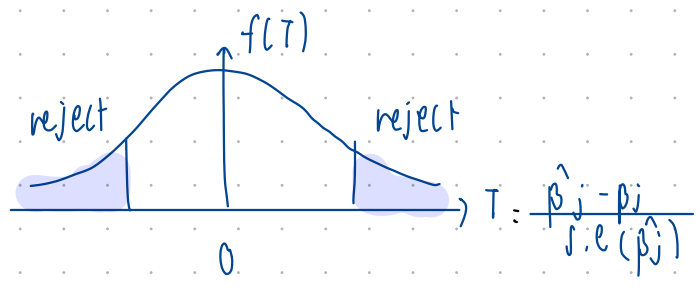
$$\text{wage} = \beta_0 + \beta_1 \text{educ} + \beta_2 \text{experience} + \dots + \epsilon$$

we want to test hypothesis about the true impact (β) of each x variable (educ, experience) on the dependent variable (y)

But we don't know what the true β are. So, we use $\hat{\beta}$ (estimator) and $\text{s.e.}(\hat{\beta})$ to test the hypothesis.



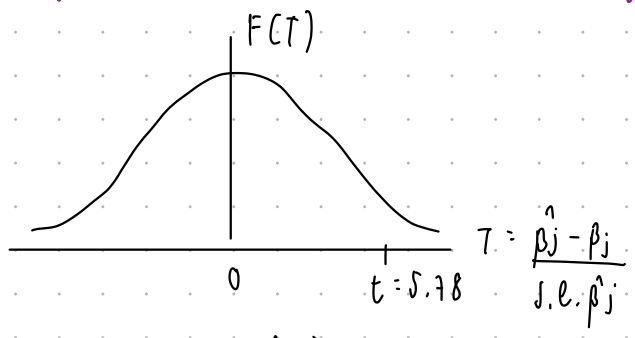
- 1) Test if $\beta =$ same number.
- eg. $\beta_j = 0 \rightarrow x_j$ has no impact on y
- $\beta_j = 1 \rightarrow 1$ unit \uparrow in x_j correspond to 1 unit \uparrow in y



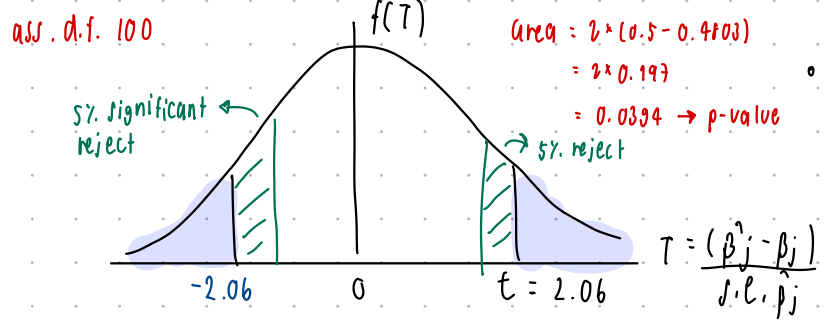
t-test how?

$$\frac{\hat{\beta}_j - \beta_j}{\text{s.e.}(\hat{\beta}_j)} \sim t.d.f.$$

Significant level = total area in the rejection region



- suppose we calculate a t-statistic
- $\frac{\hat{\beta}_j - \beta_j}{\text{s.e.}(\hat{\beta}_j)} = 2.06$
- suppose we are testing
- $H_0 : \beta_j = 0$
- $H_a : \beta_j \neq 0$ (2 tailed test)



• p-value : total shaded area

p-value = significant level which we will reject the H_0 or prob that we will reject H_0

P-value < significant level ; reject H_0

Example 1: $H_0 : \beta_j \geq 0, H_a : \beta_j < 0, d.f. = 140. \rightarrow z\text{-table}$



p-value = what should be the significant level, given the critical value of -2.75
 → find the shaded area!

⊙ 7 ⊙
 "reject"

suppose the calculated $t_{\hat{\beta}_j} = -2.75 \rightarrow t_{\hat{\beta}_j} = \frac{\hat{\beta}_j - \beta_j}{s.e.(\hat{\beta}_j)}$

From the z-table, the value -2.75 corresponds to area = 0.003

Thus, p-value = 0.003

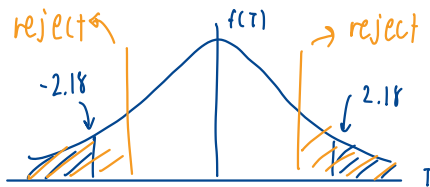
note!

⊖ ⊕ : 2 tailed
 ⊖ ⊙ : 1 tailed

Would we reject H_0 if we use the significance level = 5%?

Yes, this case is a "rule" we reject H_0 if p-value < sig. level

Example 2: $H_0 : \beta_j = a_j, H_a : \beta_j \neq a_j, d.f. = 18.$



↑ use t-table

suppose the calculated $t_{\hat{\beta}_j} = -2.18$

From the t-table, the value -2.18 corresponds to area = 0.02 to 0.05

Thus, p-value = is between 0.02 - 0.05

Would we reject H_0 if we use the significance level = 5%?

Yes, reject H_0 because the area is less than 0.05 or p-value < 0.05

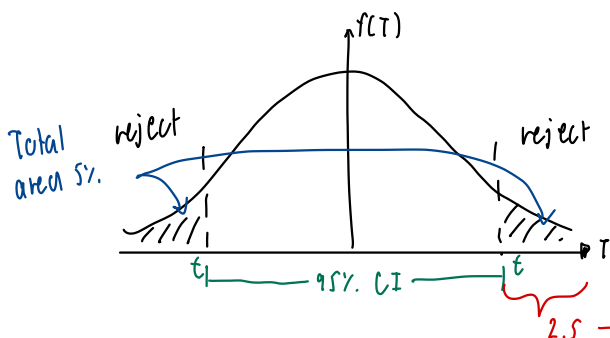
6 Confidence Intervals (CI)

Confidence Intervals for the POPULATION PARAMETER (β_j)

A 95% CI of β_j is given by ↳ The range of values that would capture the true β_j at a 5% chance

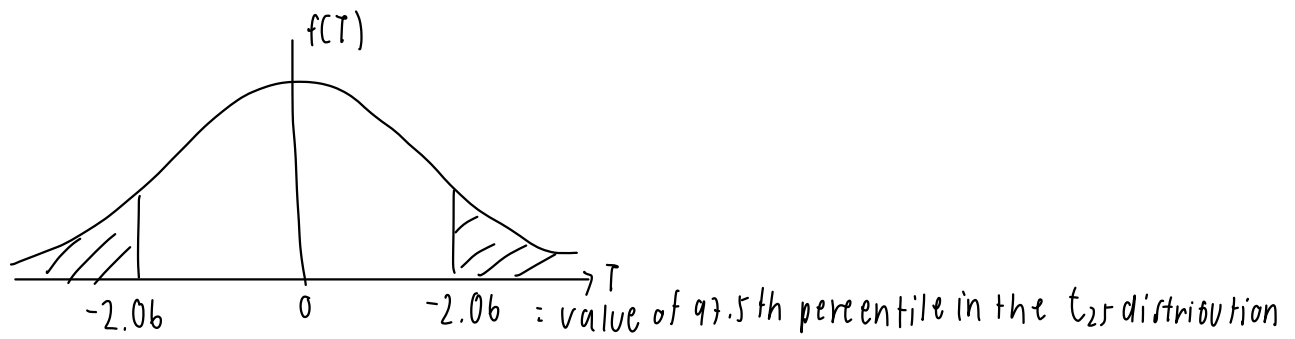
$$CI = \hat{\beta}_j \pm c \times s.e.(\hat{\beta}_j)$$

↑
 c is the 97.5 percentile in the t-distribution with $n-k-1$ d.f.



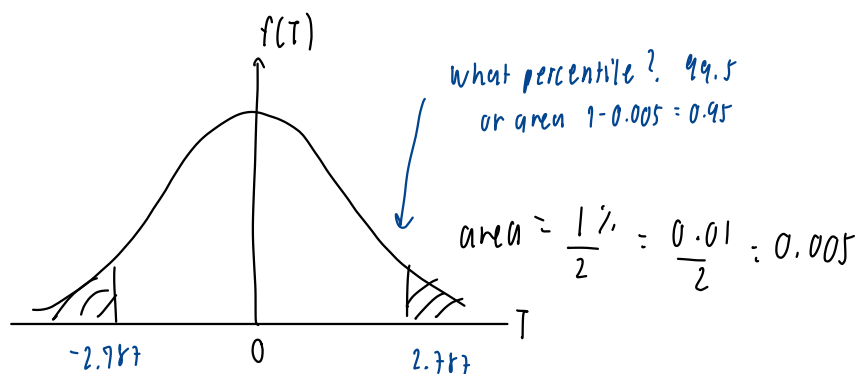
2.5 → $c = 100 - 2.5 = 97.5$

Example 1: 95% CI d.f. = 25



The 95% CI for $\hat{\beta}_j = [\hat{\beta}_j - 2.06 \cdot \text{s.e.}(\hat{\beta}_j), \hat{\beta}_j + 2.06 \cdot \text{s.e.}(\hat{\beta}_j)]$

Example 2: 99% CI d.f. = 25



The 99% CI for $\hat{\beta}_j = [\hat{\beta}_j - 2.787 \cdot \text{s.e.}(\hat{\beta}_j), \hat{\beta}_j + 2.787 \cdot \text{s.e.}(\hat{\beta}_j)]$

F-test motivation can't use t or z test as they have many factors/variables

- we want to test the significance of a group of hypothesis (multiple hypotheses)

$$\text{Grade}_{325} = \beta_0 + \beta_1 \text{no_times_front} + \beta_2 \text{no_times_back} + \beta_3 \text{hr_study} + \beta_4 \text{past_GPA} + \beta_5 \text{sender} + u$$

H_0 : seat position doesn't have impact on GPA

$$\beta_1 = 0 \text{ and } \beta_2 = 0 \text{ + } \beta_1 = \beta_2 = 0$$

H_a : seat position matters

$$\beta_1 \neq 0 \text{ and } \beta_2 \neq 0$$

$$\text{or } \beta_1 \neq 0 \text{ " } \beta_2 = 0$$

$$\text{or } \beta_1 = 0 \text{ " } \beta_2 \neq 0$$

} at least one of the $\beta_1, \beta_2 \neq 0$

7 Testing Multiple Linear Restrictions: The F-test

Suppose the model is specified by

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + u$$

$$H_0 : \beta_2 = 0 \text{ and } \beta_3 = 0 \quad \rightarrow \text{we want to test if } x_1 \text{ and } x_2 \text{ both have no impact on } y$$

$$H_a, H_1 : H_0 \text{ is not true}$$

We can use the F-test to test this type of "multiple hypotheses".

1. Our full model is called the "unrestricted" model (ur). Suppose it can be expressed as:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + u \text{ is true } \rightarrow \text{reject } H_0$$

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_3 + \dots + \beta_k x_k + u$$

2. The model which takes out x (which we think its associated $\beta = 0$) is called the restricted model (r).

$$y = \beta_0 + \beta_1 x_1 + u \text{ is true } \rightarrow \text{do not reject } H_0$$

Suppose there are q number of β that we would like to perform a joint-test of $= 0$
 eg. in above model $q = 2$

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_{k-q} x_{k-q} + u$$

$$H_0 : \beta_{k-q+1} = \beta_{k-q+2} = \dots = \beta_k = 0 \text{ (the last } q \text{ } \beta = 0)$$

$$H_a : H_0 \text{ is not true}$$

$$y = \underbrace{\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_{k-q} x_{k-q}}_{\text{restricted}} + \underbrace{\beta_{k-q+1} x_{k-q+1} + \beta_{k-q+2} x_{k-q+2} + \dots + \beta_k x_k}_{\text{unrestricted}} + u$$

$$F = \frac{(SSR_r - SSR_{ur}) / q}{SSR_{ur} / (n-k-1)}$$

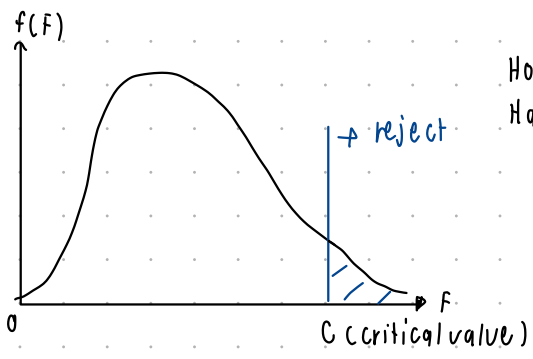
\leftarrow this is always \oplus
 b/c $SSR_{ur} < SSR_r$
 every time you add 1 more x ,
 the model would be better explained
 \uparrow df. of the "ur" model

Q: So, if every time you add 1 more x variable, the $SSR \downarrow$ and $R^2 \uparrow$, why don't we just keep the additional x in the model?

A: Every time we add 1 more x , $\text{var}(\hat{\beta}_s)$ will \uparrow , making the prediction of β less precise.

So, we only keep the addition x if it/they can improve the model enough

\hookrightarrow can \downarrow SSR ($\uparrow R^2$) enough
 or
 can significantly \downarrow SSR and $\uparrow R^2$



$H_0 : \beta_2 = \beta_3 = \dots = 0$
 $H_a : H_0 \text{ is not true}$

$F \sim F_{q, n-k-1}$
df. of ur model.
no. of joint Hypothesis being tested

We reject H_0 of jointly no effect if $F > c$
critical value
F-statistic

3. Some useful facts

- ① $R^2_{ur} > R^2_r$ b/c any additional x would $\uparrow R^2$ (improve fit)
 - $SSR_{ur} < SSR_r$
- ② By including more x , the model is certainly better explained. However, we would like to reject H_0 if the inclusion of extra variables does not improve the model enough.

4. Other ways to calculate the F-statistics:

From $R^2 = 1 - \frac{SSR}{SST} \rightarrow \frac{RSS}{TSS}$

we have $F = \frac{(R^2_{ur} - R^2_r)}{q} \rightarrow$ no. of β that are set to "0"

$\frac{(1 - R^2_{ur})}{n - k - 1 + \text{intercept}}$

\uparrow \rightarrow no. of slope β

no. of observation

If we want to test the overall significance of the model

$H_0 : \beta_1 = \beta_2 = \beta_3 = \dots = \beta_k = 0$ ↳ always "given"

$H_a : \text{otherwise}$

$F = \frac{\frac{R^2}{k}}{\frac{(1 - R^2)}{n - k - 1}}$

$\rightarrow R^2$ of the model $\approx ur$

the "r" model had no x at all

Example: Suppose we are interested in understanding the determinant of a baseball player's salary.

- \downarrow salary = season salary
- \downarrow years = years in major leagues
- \downarrow gamesyr = games per year in the league
- avg = career batting average
- hrunsyr = homeruns per year
- rbisyr = runs batted in per year

if we want to test whether performance had any impact on salary

- $H_0 : \beta_{avg} = \beta_{hrunsyr} = \beta_{rbisyr} = 0$
- $H_a : \text{otherwise is true}$

the unrestricted model (ur) is defined by

yr model

Y X

```
. regress log_salary years gamesyr bavg hrunsyr rbisyr
```

Source	SS	df	MS
Model	308.989208	5	61.7978416
Residual	183.186327	347	.527914487
Total	492.175535	352	1.39822595

Number of obs = 353
 F(5, 347) = 117.06
 Prob > F = 0.0000
 R-squared = 0.6278
 Adj R-squared = 0.6224
 Root MSE = .72658

log_salary	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]
years	.0688626	.0121145	5.68	0.000	.0450355 .0926898
gamesyr	.0125521	.0026468	4.74	0.000	.0073464 .0177578
bavg	.0009786	.0011035	0.89	0.376	-.0011918 .003149
hrunsyr	.0144295	.016057	0.90	0.369	-.0171518 .0460107
rbisyr	.0107657	.007175	1.50	0.134	-.0033462 .0248776
_cons	11.19242	.2888229	38.75	0.000	10.62435 11.76048

5 = k

v = q

Intercept

the restricted model (r) is defined by

Y X

```
. regress log_salary years gamesyr
```

Source	SS	df	MS
Model	293.864058	2	146.932029
Residual	198.311477	350	.566604221
Total	492.175535	352	1.39822595

Number of obs = 353
 F(2, 350) = 259.32
 Prob > F = 0.0000
 R-squared = 0.5971
 Adj R-squared = 0.5948
 Root MSE = .75273

log_salary	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]
years	.071318	.012505	5.70	0.000	.0467236 .0959124
gamesyr	.0201745	.0013429	15.02	0.000	.0175334 .0228156
_cons	11.2238	.108312	103.62	0.000	11.01078 11.43683

When considering each of the performance X one-by-one, none of them has a significant impact at 5%.

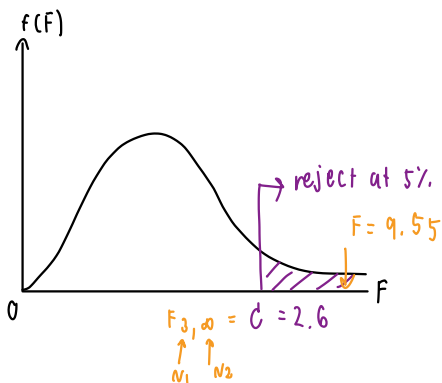
But, when performing an F-test, performances have joint impact.

Now, our H_0 and H_a becomes

$$F = \frac{\frac{SSR_r - SSR_{ur}}{q}}{\frac{SSR_{ur}}{n-k-1}} = \frac{198.311 - 183.186}{3} = \frac{15.125}{353-5-1} \approx 9.55$$

$$F = \frac{(R^2/q)}{(1-R^2)/(n-k-1)}$$

same value!



let's use 5% level of significance (from page 966 table)

∴ Since $F = 9.55 > 2.6$, we reject H_0 at 5% level and conclude that performances have joint effects on salary.

8 How the Hypothesis Testing is done in Practice

1. Check the values of t - *statistic* reported by the statistical software (i.e. STATA, SPSS, SAS)

/ These t - *statistics* are to test $H_0 : \beta_i = 0$

/ If the d.f. > 30, then when $t > \underline{1.96}$, we can reject H_0 *with 5% significance level*
 → t-table

/ **When $t > 1.96$** , we can say that β_i is **statistically significant** at 5% level. (value of $\beta_i \neq 0$)

/ **When $t < 1.96$** we can say that β_i is **not statistically significant** at 5% level.

/ If $t < 1.96$ we can drop x_i from the model

/ After we drop x_i , we estimate the new regression function and obtain a new set of $\hat{\beta}$.

2. We can also perform other hypothesis testings of interest.

e.g. $H_0 : \beta_i = \beta_j$

or $H_0 : \beta_i = 5$ etc.

or perform an F-test for testing multiple linear restrictions

3. Usually, in economics, the estimation results are reported using this form

Dependent Variable: log(salary)			
Independent Variables	(1)	(2)	(3)
log(sales)	.224 (.027)	.158 (.040)	.188 (.040)
log(mktval)	—	.112 (.050)	.100 (.049)
profmarg	—	-.0023 (.0022)	-.0022 (.0021)
ceoten	—	—	.0171 (.0055)
comten	—	—	-.0092 (.0033)
intercept	4.94 (0.20)	4.62 (0.25)	4.57 (0.25)
Observations	177	177	177
R-squared	.281	.304	.353

sales →

other company performance {

CEO characteristics {

↑
like a simple regression with 1 X

Multiple Regression Analysis : Further Issues

1 Data scaling on OLS statistics

When we change the unit of measurement of a variable, the value of estimators would change accordingly. For example

$$\widehat{bwght}_g = \hat{\beta}_0 + \hat{\beta}_1 cigs + \hat{\beta}_2 faminc,$$

where

$bwght$ = child birth weight, in grams.

$cigs$ = number of cigarettes smoked by the mother while pregnant, per day.

$faminc$ = annual family income, in thousands of dollars.

- What if we use $bwght$ in kilograms?

$$\begin{aligned} 1 \text{ kg} &= 1000 \text{ g} \\ \widehat{bwght}_{kg} &= \frac{\widehat{bwght}_g}{1000} = \frac{\hat{\beta}_0}{1000} + \frac{\hat{\beta}_1}{1000} cigs + \frac{\hat{\beta}_2}{1000} faminc \\ &= \hat{\alpha}_0 + \hat{\alpha}_1 cigs + \hat{\alpha}_2 faminc \\ \hat{\alpha}_0 &= \frac{\hat{\beta}_0}{1000}, \quad \hat{\alpha}_1 = \frac{\hat{\beta}_1}{1000}, \quad \hat{\alpha}_2 = \frac{\hat{\beta}_2}{1000} \end{aligned}$$

when using STATA

- gen $bwght_g = bwght * 28.35$
- gen $bwght_kg = bwght_g / 1000$
- regress $bwght_g$ cigs faminc
- regress $bwght_kg$ cigs faminc

from into

- what if we use $faminc$ in USD (instead of 1000 USD)

$$\begin{aligned} bwght_g &= \hat{\beta}_0 + \hat{\beta}_1 cigs + \frac{\hat{\beta}_2}{1000} faminc_{USD} \\ &= \hat{\beta}_0 + \hat{\beta}_1 cigs + \hat{\theta}_2 faminc_{USD} \\ \hat{\theta}_2 &= \frac{\hat{\beta}_2}{1000} \text{ in other words } \hat{\theta}_2 = \text{impact of 1 USD } \uparrow \text{ in income} \\ \hat{\beta}_2 &= \text{impact of 1000 USD } \uparrow \text{ in income} \end{aligned}$$

when using STATA

- gen $faminc_usd = faminc * 1000$
- regress $bwght_g$ cigs faminc
- regress $bwght_g$ cigs faminc_usd

- What if we use $bwght$ in kg and income in THB

$$\widehat{bwght}_{kg} = \frac{\hat{\beta}_0}{1000} + \frac{\hat{\beta}_1}{1000} cigs + \frac{\hat{\beta}_2}{30,000} faminc_{THB}$$

this value is going to be 30,000 times more than faminc

assume 30 THB = 1 USD

when using STATA

- gen $faminc_thb = faminc * 30,000$
- reg $bwght_g$ cigs faminc
- reg $bwght_kg$ cigs faminc_thb

2 More on functional forms

Logarithmic Functional Form

natural log

$$\log(y) = \beta_0 + \beta_1 \log(x_1) + \beta_2 x_2 + u$$

$$\Delta y = y_1 - y_2$$

$$\Delta x_1 = x_{11} - x_{12}$$

$$\beta_1 = \frac{d \log(y)}{d \log(x)} = \frac{\frac{1}{y} dy}{\frac{1}{x_1} dx_1} = \frac{\frac{1}{y} \Delta y}{\frac{1}{x_1} \Delta x_1} = \frac{100 \times \frac{1}{y} \Delta y}{100 \times \frac{1}{x_1} \Delta x_1} = \frac{\% \Delta y}{\% \Delta x}$$

with the log y and log x format, the coefficient is going to be the elasticity (x1 elasticity of y)

$$\beta_2 = \frac{d \log(y)}{dx_2} = \frac{\frac{1}{y} dy}{dx_2} = \frac{\frac{1}{y} \Delta y}{\Delta x_2}$$

if we want the upper term to be % change, $100 \beta_2 = \% \Delta$ in y given that x_2 increased by 1 unit then,

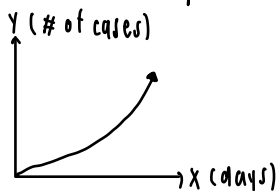
$$100 \beta_2 = \frac{100 \times \frac{1}{y} \Delta y}{\Delta x_2}$$

$$100 \beta_2 = \frac{\% \Delta y}{\Delta x_2}$$

Models with Quadratics (Squares)

• Capture increasing / decreasing marginal effects (slope of the relationship between x and y is not constant)

Covid-19 example

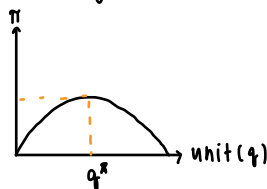


$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + u$$

$$\frac{dy}{dx} = \beta_1 + 2\beta_2 x$$

↑ ⊕ ↑ ⊕ ↘ days

Decreasing returns



$$y = \beta_0 + \beta_1 x + \beta_2 x^2 + u$$

$$\frac{dy}{dx} = \beta_1 + 2\beta_2 x$$

↑ ⊕ ↑ ⊖

$$\pi = (p - mc)q; \quad mc = 100, \quad \text{demand: } p = 100 - q$$

$$\pi = (100 - q - 10)q$$

$$\text{F.O.C. } \frac{\partial \pi}{\partial q} = 0 = 90 - 2q$$

↖ β1 ⊕
↘ β2 ⊖

Example : Effects of Pollution on Housing Prices

$$\log(\text{price}) = \beta_0 + \beta_1 \log(\text{nox}) + \beta_2 \log(\text{dist}) + \beta_3 \text{rooms} + \beta_4 \text{room}^2 + \beta_5 \text{stratio} + u$$

Note : $\frac{d}{dx}(\ln x) = \frac{1}{x}$
 $d \ln(x) = \frac{1}{x} dx$

where

- price = housing price
 - nox = level of pollution
 - dist = distance from downtown
 - rooms = number of rooms
 - stratio = average student per teacher ratio
- The estimation result is given by

In the US or many other countries, students can apply to schools in the area without having to take any test. So, the lower stratio, the better the school.

regress lprice lnox dist rooms rooms_sq stratio

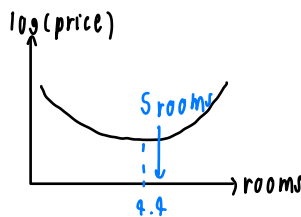
Source	SS	df	MS			
Model	51.4933152	5	10.298663	Number of obs =	506	
Residual	33.0889098	500	.06617782	F(5, 500) =	155.62	
Total	84.582225	505	.167489554	Prob > F =	0.0000	
				R-squared =	0.6088	
				Adj R-squared =	0.6049	
				Root MSE =	.25725	

	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
log(price) → lprice						
log(nox) → lnox	β_1 -.9767545	.0995938	-9.81	0.000	-1.172429	-.7810806
dist	β_2 -.0321972	.0094013	-3.42	0.001	-.050668	-.0137264
rooms	β_3 -.5528032	.1612965	-3.43	0.001	-.8697056	-.2359007
rooms_sq	β_4 .0624697	.0124867	5.00	0.000	.0379368	.0870025
stratio	β_5 -.0486667	.0058131	-8.37	0.000	-.0600879	-.0372455
_cons	13.59154	.5650901	24.05	0.000	12.4813	14.70178

$|t| > 1.96$ \uparrow \uparrow all < 0.05
 all variables are significant

Consider the effect of "room"

$$\frac{d \log(\text{price})}{d \text{rooms}} = \beta_3 + 2\beta_4 \text{rooms} = -0.553 + 2(0.062) \cdot \text{rooms}$$



at how many rooms does 1 additional room has a 0 impact on log(price)?

$$0 = -0.553 + 2(0.062) \cdot \text{rooms}$$

$$\text{rooms} = 4.4$$

Answer → at 4.4 rooms or more
 at 5 rooms or more

What would be the % change in price when the number of room increases from 5 to 6?

$$\frac{d \log(\text{price})}{d \text{rooms}} = -0.553 + 2(0.062) \cdot \text{rooms}$$

$$100 \times \frac{1}{\text{price}} \frac{d \text{price}}{d \text{rooms}} = 100 (-0.553 + 2(0.062)) \cdot 5$$

$$= 100 \times 0.067$$

$$= 6.7\% \text{ increase}$$

∴ Total % ∆ in price when # rooms ↑ from 5 to 6 is 6.7 + 19.1% is 25.8%

what about % in price when # rooms increase from 5 to 6?

$$\% \Delta \text{price} = 100 (-0.553 + 2(0.062) \cdot 6)$$

$$= 19.1\% \text{ increase}$$

3 Models with Interaction Terms + used when the impact of one variable depends on the value (level) of another variable

Consider

$$price = \beta_0 + \beta_1 \underset{x_1}{sqr\ ft} + \beta_2 \underset{x_2}{bdrms} + \beta_3 \overset{x_3}{\underbrace{sqr\ ft \cdot bdrms}} + \beta_4 \underset{x_2}{bthrms} + u$$

where

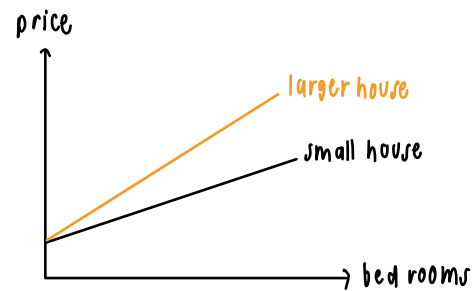
price = housing price

sqr ft = house size (square feet)

bdrms = number of bedrooms

bthrms = number of bathrooms

$$\bullet \frac{d\ price}{d\ bdrms} = \beta_2 + \beta_3 \text{ sqrft}$$



if $\beta_2 > 0$ then, an additional bedroom would increase price more for a larger house

4 More on the Goodness-of-Fit and Selection of Regressors

Adding more regressors **ALWAYS** improve fit $\uparrow R^2$ always \uparrow

• But we lose the "degree of freedom"

1 data point is sacrificed every time we estimate a parameter

• using R^2 would not punish "having too many regressors"

• we use adjusted- R^2 or \bar{R}^2 when we want to punish adding too many regressors

$$R^2 = 1 - \frac{SSR}{SST} = 1 - \frac{SSR/n}{SST/n}$$

$$\text{adj } R^2 = \frac{1 - SSR/(n-k-1)}{SST/(n-1)}$$

if we have more k , d.f. = $n-k-1 \downarrow$

$$, \quad \frac{SSR}{n-k-1} \downarrow$$

$$, \quad \text{adj-}R^2 \downarrow$$

Using adjusted R-squared to choose between non-nested models (one model is not a subset of another).

Consider Model 1

$$\begin{aligned} \widehat{\text{salary}} &= 830.63 + 0.0163\text{sales} + 19.63\text{roe} \\ & \quad (223.90) \quad (0.0089) \quad (11.08) \\ n &= 209, \quad R^2 = 0.029, \quad \bar{R}^2 = 0.020 \end{aligned}$$

Consider Model 2

$$\begin{aligned} \log(\widehat{\text{salary}}) &= 4.36 + 0.2751 \log(\text{sales}) + 0.0179\text{roe} \\ & \quad (0.29) \quad (0.033) \quad (0.004) \\ n &= 209, \quad R^2 = 0.282, \quad \bar{R}^2 = 0.275 \end{aligned}$$

↳ 27.5% of variation in y is explained
so, the model is better

Multiple Regression Analysis with Qualitative Information:

1 Outline

- # Describing qualitative information
- # Using a single dummy independent variable
- # Using dummy variables for multiple categories

- # Interactions involving dummy variables
- # A binary dependent variable (Y variable): The linear probability model

2 Describing Qualitative Information

- # "Female" and "Married" are qualitative variable.
- # We arbitrarily assign a dummy variable to describe them.

$$\begin{aligned}
 \textit{female} &= \begin{cases} 1 & \text{if female} \\ 0 & \text{otherwise (or if male)} \end{cases} \\
 \textit{married} &= \begin{cases} 1 & \text{if married} \\ 0 & \text{otherwise (of if single)} \end{cases}
 \end{aligned}$$

TABLE 7.1

A Partial Listing of the Data in WAGE1.RAW

<i>person</i>	<i>wage</i>	<i>educ</i>	<i>exper</i>	<i>female</i>	<i>married</i>
1	3.10	11	2	1	0
2	3.24	12	22	1	1
3	3.00	11	2	0	0
4	6.00	8	44	0	1
5	5.30	12	7	0	1
⋮	⋮	⋮	⋮	⋮	⋮
⋮	⋮	⋮	⋮	⋮	⋮
525	11.56	16	5	0	1
526	3.50	14	5	1	0

3 Models with a single dummy independent variable

Consider

$$wage = \beta_0 + \delta_0 female + \beta_1 educ + u.$$

where

$$female = \begin{cases} 1 & \text{if female} \\ 0 & \text{otherwise (or if male)} \end{cases}$$

In this case, the δ_0 notation is used to highlight the interpretation of the parameters multiplying dummy variables. In other cases, we can use any notation that is the most convenient.

$$\begin{aligned} E(wage | female, educ) &= E(\beta_0 + \delta_0 female + \beta_1 educ + u | female, educ) \\ &= \beta_0 + \delta_0 female + \beta_1 educ + E(u | female, educ) \\ &= \beta_0 + \delta_0 female + \beta_1 educ \end{aligned}$$

// 0 from u.s.s. MLR 1-4 holds

Thus,

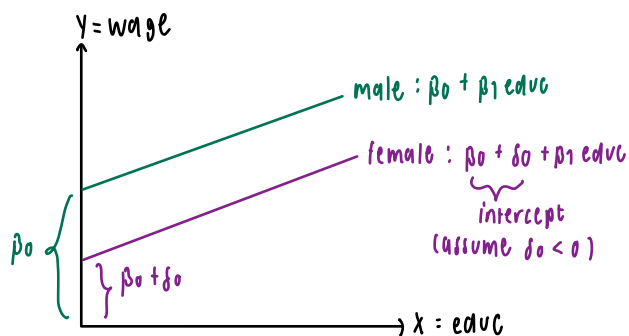
$$\text{female} \rightarrow \text{♀} : E(wage | female = 1, educ) = \beta_0 + \delta_0 (1) + \beta_1 educ = \beta_0 + \delta_0 + \beta_1 educ$$

$$\text{male} \rightarrow \text{♂} : E(wage | female = 0, educ) = \beta_0 + \delta_0 (0) + \beta_1 educ = \beta_0 + \beta_1 educ$$

$$\delta_0 = E(wage | female = 1, educ) - E(wage | female = 0, educ)$$

$$\text{or } \delta_0 = E(wage | female, educ) - E(wage | male, educ)$$

Given the same value of $educ$ (same education level), δ_0 is the difference in the expected wage of females and males



\therefore female is going to give a constant impact on wage, regardless of the level of $educ$.

4 It is not possible to include all of the dummy alternatives in the same model (as long as there is an intercept in the model)

If we include all alternatives of a dummy variable in the same model, we will face the "perfect collinearity" problem.

$$\text{wage} = \beta_0 x_0 + \beta_1 \text{female} + \beta_2 \text{educ} + \beta_3 \text{male} + u$$

For example:

↓
intercept

$$\begin{aligned} x_0 &= x_1 + x_3 \\ 1 &= \text{female} + \text{male} \\ \text{female} &= \text{male} + 1 \end{aligned}$$

or

if there are "n" categories, we omit "1" category to avoid multi collinearity

$$\begin{aligned} \textcircled{1} &= \text{winter} + \text{spring} + \text{summer} + \text{fall} \\ \text{winter} &= 1 - \text{spring} - \text{summer} - \text{fall} \end{aligned}$$

$$\text{winter} = \begin{cases} 1 & \text{if winter} \\ 0 & \text{otherwise} \end{cases}$$

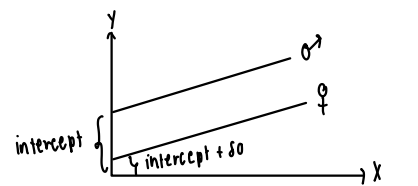
$$\text{spring} = \begin{cases} 1 & \text{if spring} \\ 0 & \text{otherwise} \end{cases}$$

etc.

id	winter	spring	summer	fall	x ₀
1	1	0	0	0	1
2	1	0	0	0	1
3	0	0	1	0	1
4	0	0	1	0	1
.	0	0	0	0	1
.	0	1	0	0	1
.	0	1	0	0	1

At least one alternative has to be dropped. We treat the dropped alternative as the "BASE GROUP" or "BASELINE" or "BENCHMARK GROUP".

```
. regress lwage female male married educ exper
note: male omitted because of collinearity
```



Number of obs = 526
 F(4, 521) = 75.27
 Prob > F = 0.0000
 R-squared = 0.3663
 Adj R-squared = 0.3614
 Root MSE = .42477

Source	SS	df	MS
Model	54.3265253	4	13.5816313
Residual	94.0032262	521	.180428457
Total	148.329751	525	.28253286

lwage	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]
female	-.3251146	.0377061	-8.62	0.000	-.3991892 -.25104
male	0	(omitted)			
married	.1380145	.0411197	3.36	0.001	.0572338 .2187953
educ	.0872644	.0071554	12.20	0.000	.0732075 .1013213
exper	.0076213	.0015314	4.98	0.000	.0046129 .0106297
_cons	.4690918	.1040575	4.51	0.000	.264668 .6735156

female workers are expected to have less wage compared to male workers

5 Using dummy variables for multiple categories

Case 1 We can use many dummy variables in the same model

Consider a model which includes 2 dummy variables— *female* and *married*.

$$\begin{array}{l} \uparrow \left\{ \begin{array}{l} 1 \text{ if female} \\ 0 \text{ otherwise} \end{array} \right. \quad \uparrow \left\{ \begin{array}{l} 1 \text{ if married} \\ 0 \text{ otherwise} \end{array} \right. \end{array}$$

$$\log(\text{wage}) = \beta_0 + \delta_0 \text{female} + \delta_1 \text{married} + \beta_1 \text{educ} + \beta_2 \text{exper} + \beta_3 \text{exper}^2 + \beta_4 \text{tenure} + \beta_5 \text{tenure}^2 + u.$$

```
regress lwage female married educ exper expersq tenure tenursq
```

Source	SS	df	MS	Number of obs = 526		
Model	65.6482326	7	9.37831895	F(7, 518) = 58.76		
Residual	82.6815188	518	.159616832	Prob > F = 0.0000		
				R-squared = 0.4426		
				Adj R-squared = 0.4351		
Total	148.329751	525	.28253286	Root MSE = .39952		

lwage	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
female	-.2901838	.0361121	-8.04	0.000	-.3611279	-.2192396
married	.0529219	.0407561	1.30	0.195	-.0271456	.1329894
educ	.0791547	.0068003	11.64	0.000	.0657952	.0925143
exper	.0269535	.0053258	5.06	0.000	.0164907	.0374163
expersq	-.0005399	.0001122	-4.81	0.000	-.0007603	-.0003196
tenure	.0312962	.0068482	4.57	0.000	.0178426	.0447499
tenursq	-.0005744	.0002347	-2.45	0.015	-.0010355	-.0001134
_cons	.4177837	.0988662	4.23	0.000	.2235557	.6120116

δ_1 measures the impact of be married. (marriage premium)

But since $|t| < 1.96$ or $p > 0.05$, we don't reject H_0 of no impact.

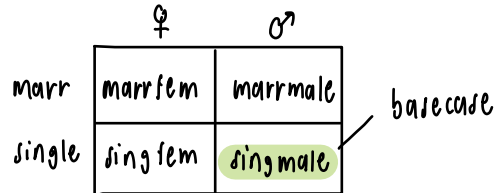
1) δ_0 measures the expected difference between female & male workers given the same marital status and other factors

$$\frac{d \log(\text{wage})}{d \text{female}} = \frac{\frac{1}{\text{wage}} d \text{wage}}{d \text{female}} = -0.29$$

$$\frac{100 \cdot \frac{1}{\text{wage}} d \text{wage}}{d \text{female}} = 100 \cdot (-0.29)$$

$$\frac{\% \Delta \text{wage}}{d \text{wage}} = 29.02 \%$$

\therefore female workers are expected to earn less than male workers by 29.02%, holding other factors the same



8. Multiple Regression Analysis with Qualitative Information: 85

Consider a model which includes dummy variables for each gender/marital status combination—*marrmale*, *marrfem* and *singfem*. (or *singmale* ← used as the basecase.)

$$\log(\text{wage}) = \beta_0 + \delta_0 \text{marrmale} + \delta_1 \text{marrfem} + \delta_3 \text{singfem} + \beta_1 \text{educ} + \beta_2 \text{exper} + \beta_3 \text{exper}^2 + \beta_4 \text{tenure} + \beta_5 \text{tenure}^2 + u. \quad (8.1)$$

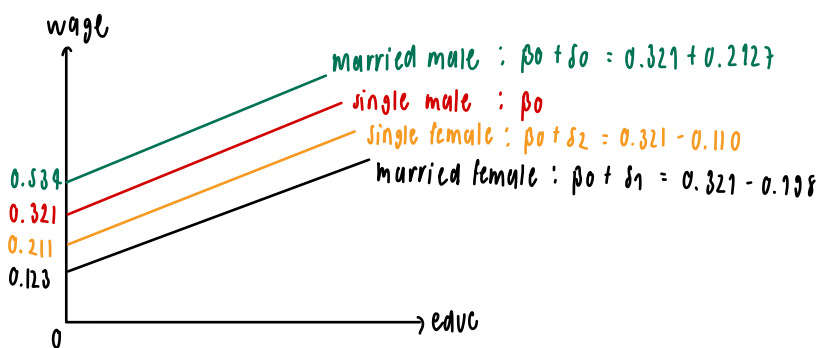
```
regress lwage marrmale marrfem singfem educ exper expersq tenure tenursq
```

Source	SS	df	MS	Number of obs = 526		
Model	68.3617623	8	8.54522029	F(8, 517) =	55.25	
Residual	79.9679891	517	.154676961	Prob > F =	0.0000	
Total	148.329751	525	.28253286	R-squared =	0.4609	
				Adj R-squared =	0.4525	
				Root MSE =	.39329	

lwage	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]	
δ_0 marrmale	.2126757	.0553572	3.84	0.000	.103923	.3214284
δ_1 marrfem	-.1982676	.0578355	-3.43	0.001	-.311889	-.0846462
δ_2 singfem	-.1103502	.0557421	-1.98	0.048	-.219859	-.0008414
educ	.0789103	.0066945	11.79	0.000	.0657585	.092062
exper	.0268006	.0052428	5.11	0.000	.0165007	.0371005
expersq	-.0005352	.0001104	-4.85	0.000	-.0007522	-.0003183
tenure	.0290875	.006762	4.30	0.000	.0158031	.0423719
tenursq	-.0005331	.0002312	-2.31	0.022	-.0009874	-.0000789
_cons	.3213781	.100009	3.21	0.001	.1249041	.5178521

This regression is not the same as the previous one. It uses "single male" as the base group. (the previous one use male & single as 2 base groups)

- ① δ_0 measures the expected diff. in wage of married male as compared with single males, holding other factors constant.
- ② δ_1 measures the expected diff. in wage of married female as compared with single males, holding other factors constant.
- ③ δ_2 same rationale



Case 2 We can use dummy variables to represent multiple categories of a variable
 Consider the relationship between law school rankings and starting salaries

$$\log(\text{salary}) = \beta_0 + \delta_0 \text{top10} + \delta_1 r11_25 + \delta_3 r26_40 + \delta_4 r41_60 + \beta_1 \text{LSAT} + \beta_2 \text{GPA} + \beta_3 \log(\text{libvol}) + \beta_4 \log(\text{cost}) + u.$$

where *top10*, *r11_25*, *r26_40*, *r41_60* would be equal to 1 when the variable *rank* falls into the appropriate range.

** Rank below 60 would be the base case.

In many cases the "range of value" serve as a better explanatory variable than the "value" itself. eg age may explain the model better if split into generations young 0-15, gen 2 16-29, etc.

```
. regress lsalary top10 r11_25 r26_40 r41_60 LSAT GPA llibvol lcost
```

Source	SS	df	MS	Number of obs =	136
Model	9.16538532	8	1.14567316	F(8, 127) =	120.15
Residual	1.2109665	127	.009535169	Prob > F =	0.0000
Total	10.3763518	135	.076861865	R-squared =	0.8833
				Adj R-squared =	0.8759
				Root MSE =	.09765

lsalary	Coef.	Std. Err.	t	P> t	[95% Conf. Interval]
top10	.5393428	.053542	10.07	0.000	.4333927 .6452928
r11_25	.4716199	.0390921	12.06	0.000	.3942637 .548976
r26_40	.2790977	.0346972	8.04	0.000	.2104383 .3477571
r41_60	.182382	.0283098	6.44	0.000	.126362 .238402
LSAT	.0060482	.0034919	1.73	0.086	-.0008616 .012958
GPA	.1305893	.0818678	1.60	0.113	-.0314122 .2925908
llibvol	.0725522	.0289213	2.51	0.013	.0153221 .1297824
lcost	.0249169	.0283224	0.88	0.381	-.031128 .0809619
_cons	8.363103	.4457314	18.76	0.000	7.481081 9.245125

the baseline is ranking 61th and worse

- ① δ_0 measures the difference in expected $\log(\text{salary})$ of a law-school graduate from a top 10 university compared to expected $\log(\text{salary})$ of those who graduated from the school ranked 61th and worse
- ② δ_1 use the same rationale