

Solving System of Linear Equations

TU152: Fundamental Mathematics

Saifon Chaturantabut

Department of Mathematics and Statistics
TU

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Linear equations

We consider a linear equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where a_1, a_2, \dots, a_n are *constants* and x_1, x_2, \dots, x_n are *variables* or *unknowns*.

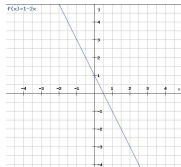
Notice, this is *linear* in term of the *power* of $x_i, i = 1, \dots, n$.

- The variables x_1, x_2, \dots, x_n that solve or satisfy the equation is called **solution** of that equation.
- This equation generally has infinitely many solutions when $n \geq 2$.
- **Example:** for $n = 2$, the equation

$$2x + y = 1$$

has *infinitely many solutions* for (x, y) .

- Two of the solutions are: $x = 1, y = -1$ and $x = 0, y = 1$
- Set of all solutions: $\{(x, y) = (a, 1 - 2a) | a \in \mathbb{R}\}$, which are on the a straight line: $y = 1 - 2x$.



- In general, if the number of equations is less than the number of variables, there could be infinite many solutions.

System of linear equations

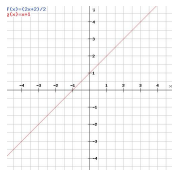
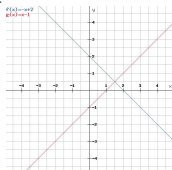
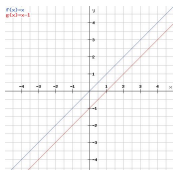
The general form of system of linear equations with n equations and n unknowns is:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ &\vdots \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

where a_{ij} and b_i are constant real numbers, for $i, j = 1, 2, \dots, n$; x_1, \dots, x_n are variables or unknowns.

There are 3 possible cases for the solution of a system of linear equations with n equations and n unknowns.

- There is no solution. E.g. $\begin{cases} -x + y = 0 \\ -x + y = -1 \end{cases}$
- There is exactly one unique solution. E.g. $\begin{cases} x + y = 2 \\ -x + y = -1 \end{cases}$
- There are many solutions. E.g. $\begin{cases} -2x + 2y = 2 \\ -x + y = 1 \end{cases}$



System of linear equations

- The general form of system of linear equations is:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots \quad \vdots \quad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

where a_{ij} and b_i are some constant numbers, for $i, j = 1, 2, \dots, n$.

- This system can be written in the matrix-vector form as:

$$\mathbf{Ax} = \mathbf{b} \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

- Augmented matrix of the above linear system is given by

$$[A|b] \quad \text{or} \quad \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array} \right].$$

Example Equivalent forms of linear system.

$$\left. \begin{array}{l} (R_1) \quad a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ (R_2) \quad a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ (R_3) \quad a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{array} \right\} \Leftrightarrow \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array} \right]$$

Example: Write the following linear system in matrix-vector form.

$$\begin{array}{rclclcl} x_1 & + & x_2 & + & x_3 & = & 1 \\ x_1 & + & 2x_2 & + & 3x_3 & = & 2 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & 1 \end{array}$$

Transforming system of linear equations: Elementary row operations

Elementary row operations are the following three rules for manipulating or transforming an augmented matrix that leave the values of the solution set unchanged.

Elementary row operations

1. Any two rows can be exchanged.
2. Any row may be multiplied (or divided) by a non-zero constant.
3. A multiple of any row can be added to any other row.

Elementary row operations are used in the **Gaussian elimination method** to transform the linear systems into the “**upper triangular form**”, which can be used to obtain the solution easily.

Example: The following linear systems have the same solution $\{x_1, x_2, x_3\}$.

$$\begin{aligned}
 \text{(i)} & \left\{ \begin{array}{l} (R_1) : x_1 + x_2 + x_3 = 1 \\ (R_2) : x_1 + 2x_2 + 3x_3 = 2 \\ (R_3) : 4x_1 + 5x_2 + 6x_3 = 1 \end{array} \right\} \\
 \text{(ii)} & \left\{ \begin{array}{l} (\bar{R}_1) : 2x_1 + 2x_2 + 2x_3 = 2 \\ (\bar{R}_2) : 0.5x_1 + x_2 + 1.5x_3 = 1 \\ (\bar{R}_3) : 4x_1 + 5x_2 + 6x_3 = 1 \end{array} \right\} (\bar{R}_1) = 2(R_1), (\bar{R}_2) = \frac{1}{2}(R_2) \\
 \text{(iii)} & \left\{ \begin{array}{l} (\hat{R}_1) : x_1 + x_2 + x_3 = 1 \\ (\hat{R}_2) : x_2 + 2x_3 = 1 \\ (\hat{R}_3) : 6x_1 + 7x_2 + 8x_3 = 3 \end{array} \right\} (\hat{R}_2) = (R_2) - (R_1), (\hat{R}_3) = (R_3) + 2(R_1)
 \end{aligned}$$

Gaussian Elimination Method

Gaussian elimination method is a systematic way for solving a linear system. This method (with some modification) is generally well suited for solving *large-scale* linear systems on a computer. In particular, it is used for solving linear systems in the form:

$$\mathbf{Ax} = \mathbf{b}$$

by performing two procedures:

- I. **Forward elimination** to transform the system into the *upper diagonal form*:

$$[\mathbf{A} \quad \mathbf{b}] \Rightarrow [\tilde{\mathbf{A}} \quad \tilde{\mathbf{b}}]$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\ a_{31} & a_{23} & a_{33} & \dots & a_{3n} & b_3 \\ \vdots & & & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & & a_{nn} & b_n \end{array} \right] \Rightarrow \left[\begin{array}{cccc|c} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} & \dots & \tilde{a}_{1n} & \tilde{b}_1 \\ 0 & \tilde{a}_{22} & \tilde{a}_{23} & \dots & \tilde{a}_{2n} & \tilde{b}_2 \\ 0 & 0 & \tilde{a}_{33} & \dots & \tilde{a}_{3n} & \tilde{b}_3 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \tilde{a}_{nn} & \tilde{b}_n \end{array} \right]$$

- II. **Back substitution** to solve for the unknowns $\mathbf{x} = [x_1, \dots, x_n]^T$

- $x_n = \frac{b_n}{a_{nn}}$

- $x_i = \frac{1}{a_{ii}} (b_i - \sum_{k=i+1}^n a_{ik} x_k)$ for $i = n-1, n-2, \dots, 2, 1$.

Gaussian Elimination Method

Example: Solve the following system of linear equations by using Gaussian elimination method.

$$\begin{aligned}x - y &= 2 \\2x - y - z &= 3 \\x + y + z &= 6\end{aligned}$$

Answer: The corresponding augmented matrix of the given linear system can be written as

$$\begin{aligned}R_1 : & \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \end{array} \right] \\R_2 : & \left[\begin{array}{ccc|c} 2 & -1 & -1 & 3 \end{array} \right] \\R_3 : & \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \end{array} \right] \end{aligned} \Leftrightarrow \begin{aligned}x - y &= 2 \\2x - y - z &= 3 \\x + y + z &= 6\end{aligned}$$

and we will apply the following 2 procedures in Gaussian elimination to the augmented matrix.

- I. Forward elimination to obtain the upper triangular form
- II. Back substitution

Gaussian Elimination Method: Example

I. Forward elimination

$$\begin{array}{l} R_1 : \\ R_2 : \\ R_3 : \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 2 & -1 & -1 & 3 \\ 1 & 1 & 1 & 6 \end{array} \right] \Leftrightarrow \begin{array}{r} x - y = 2 \\ 2x - y - z = 3 \\ x + y + z = 6 \end{array}$$

Step 1: R_1 is the "pivot row" and $a_{11} = 1$ is the "pivot element".

Let $m_{21} = \frac{a_{21}}{a_{11}} = \frac{2}{1} = 2$ and $m_{31} = \frac{a_{31}}{a_{11}} = \frac{1}{1} = 1$.

$$\begin{array}{l} R_1 : \\ R_2 \mapsto R_2 - m_{21}R_1 : \\ R_3 \mapsto R_3 - m_{31}R_1 : \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & 1 & 4 \end{array} \right] \Leftrightarrow \begin{array}{r} x - y = 2 \\ y - z = -1 \\ 2y + z = 4 \end{array}$$

Step 2: R_2 is the "pivot row" and $a_{22} = 1$ is the "pivot element".

Let $m_{32} = \frac{a_{32}}{a_{22}} = \frac{2}{1} = 2$.

$$\begin{array}{l} R_1 : \\ R_2 : \\ R_3 \mapsto R_3 - m_{32}R_2 : \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 3 & 6 \end{array} \right] \Leftrightarrow \begin{array}{r} x - y = 2 \\ y - z = -1 \\ 3z = 6 \end{array}$$

II. Back substitution

- Row R_3 : $z = \frac{6}{3} = 2$
- Row R_2 : $y = -1 + z = -1 + 2 = 1$
- Row R_1 : $x = 2 + y = 3$

That is, the solution: $x = 3, y = 1, z = 2$. ■

Gaussian Elimination Method: Example

Example: Solve the following system of linear equations by using Gaussian elimination method.

$$\begin{array}{rcccccc} 2x & + & 8y & + & 4z & = & 2 \\ 2x & + & 5y & + & z & = & 5 \\ 4x & + & 10y & - & z & = & 1 \end{array}$$

Example (continued):

Gaussian Elimination Method: Example - no solution

Example: Solve the following system of linear equations by using Gaussian elimination method.

$$\begin{array}{rcccccc} x & + & 2y & + & 3z & = & 0 \\ 4x & + & 5y & + & 6z & = & 3 \\ 7x & + & 8y & + & 9z & = & 0 \end{array}$$

Example (continued):

Gaussian Elimination Method: Example - many solutions

Example: Solve the following system of linear equations by using Gaussian elimination method.

$$2x + 4y + 6z = 0$$

$$4x + 5y + 6z = 3$$

$$7x + 8y + 9z = 0$$

Example (continued):