LECTURE ON INTRODUCTORY MATHEMATICAL ECONOMICS

SEMESTER 2/2021

Asst. Prof. Kittichai Saelee, Ph.D.

Table of Contents

Chapter 1: Nature of economics analysis and Element of mathematical economics model

1.1 Structure of economic analysis

- How do we study economics?
  - *Model-based analysis*
- What is model?
  - Model is an *abstract* form of the problem that we are interested in.
  - The problem that we (economists) are interested in aims at getting us some insights, *helping us understand something about how the world/economy works.*
  - Typically, all the results in a certain model hinges on the *assumptions* used in the analysis.
- Purpose of economics model; they are all developed to answer three broad classes of analytical questions in economics;
  - *Static equilibrium analysis*
  - *Comparative static analysis*
  - *Dynamic analysis*

Example 1.A: *The Basic 101 Model, The market mechanism model!*

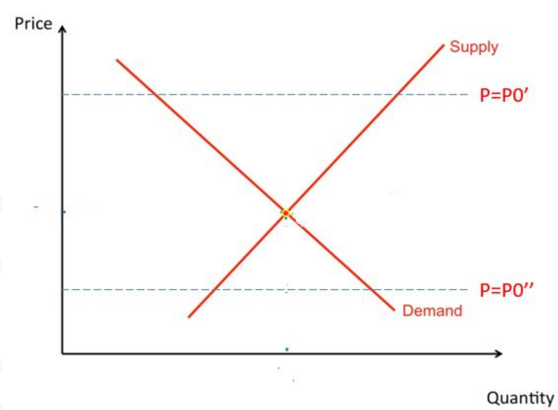
Question: At what price and quantity of output, will the *buyers and sellers* trade in the market?

- To understand the trading outcome, economists model the interaction of the two trading parties in the market using the basic **demand and supply model**.
  - The model basically builds upon several assumptions.
- Conventionally, two behavioral *assumptions* are that
  - **Demand:** Given *all other factors*, quantity demanded is negatively related to price
    - *Downward sloping curve*
  - **Supply:** Given *all other factors*, quantity supplied is positively related to price.
    - *Upward sloping curve*

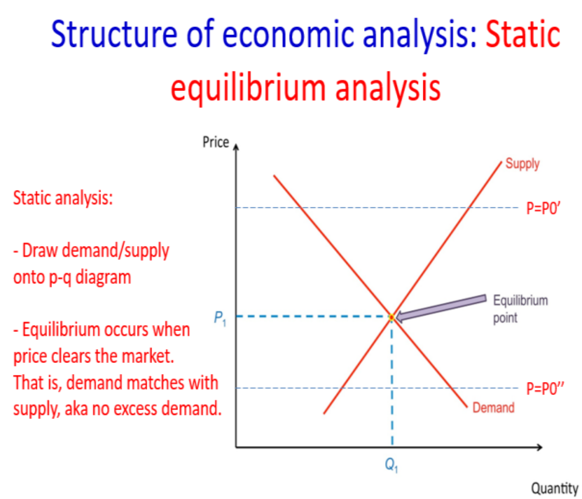
- Some other **implicit assumptions** include (i) *agents are allowed to adjust their price freely*, (ii) *trading can repeatedly occur over times under a decentralized system.*

To see how they interact and the way outcome is determined, the model is mostly represented with the aid of diagram.

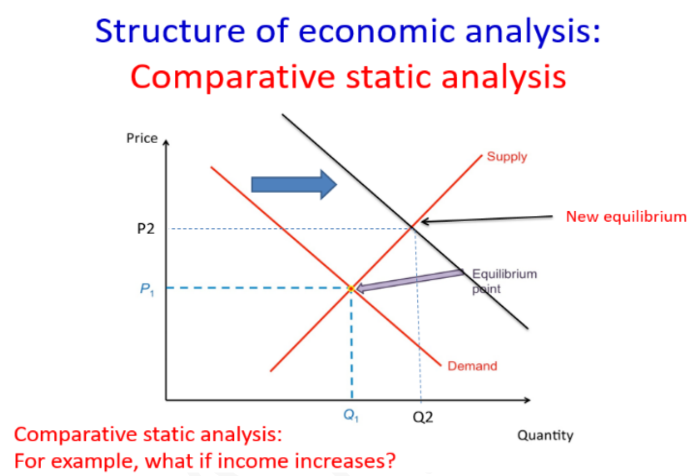
- If P = P0’ and P = P0’’, what are the trading outcome under each respective cases?
    - Obviously, through the repeated interaction, neither prices represent the steady outcome emerged under repeatedly trading environment.



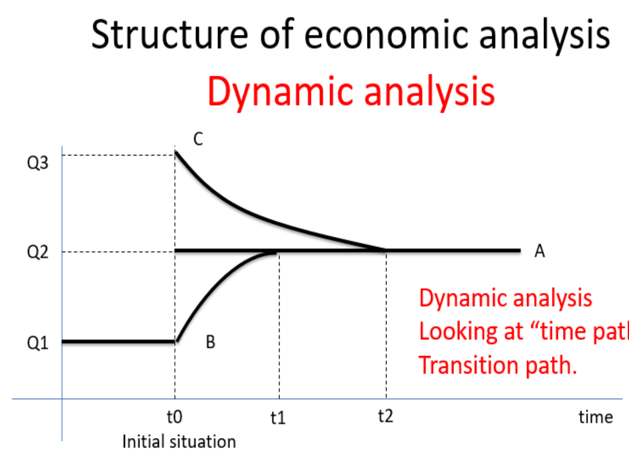
- Economists aim to uncover the equilibrium! The notion of equilibrium is then defined as a *sustained* outcome with *no further adjustments*: *Static equilibrium analysis*
    - With the aid of diagram, the equilibrium is illustrated by the intersection between “demand” and “supply” curve.



- Economists typically ask further about what would happen to the equilibrium if all those presumed existing conditions turn out to be varying: *comparative static analysis.*



- In many cases, economists are interested in seeing the adjustment process over time, i.e. transitional process. This issue is related to "*dynamic analysis*".



Where we are headed in this semester is to formalize all these analytical questions by using *mathematical tools.* There are at least two (important) reasons why we reformulate the whole economic concepts/problems/questions in terms of mathematics.

- First, mathematics is more general approach to represent ideas. Therefore, this thus allows us to easily address/conceptualize the problem in *highe- or multi-dimensional analysis*.

- Second, economists need to deal with empirical analysis, i.e. validate the model, and *cross-check the model implications with data.*
    - While qualitative predictions are important, many are more interested in quantitative ones.
    - This fundamentally requires the formulation of a model in its quantitative version; mathematics evidently come into play.
    - In addition, statistical techniques can be linked to mathematical models, allowing for a more rigorous validation of model against the data.

1.2 Economics model in Math framework

Formulating economic concepts in terms of math requires the representation.

- Variables
    - Quantity or amount of some objects whose value can be changed.
    - For example, Y is a variable that represents "your income in a day". X is another variable that represents how many hours you have worked in a day. (The value of both Y and X can be changed.)

- Equation
    - An equation describes the relationship of variables, at the least two.
    - How Y and X can be linked together is summarized by an equation.
    - For example, $Y = 9X$ if hourly wage (w) is \$9.

In general, economics usually discusses about the *interrelationship / interlinkage* among variables, and how the interaction brings about the so called "*equilibrium*". Therefore, most economics model are best represented by a class of mathematical system so called "*system of simultaneous equation(s)*". The system of simultaneous equations takes the following representation,

$$\begin{aligned} a_{11}x_1 + \ & a_{12}x_2 + \cdots + \ a_{1n}x_n = b_1 \\ a_{21}x_1 + \ & a_{22}x_2 + \cdots + \ a_{2n}x_n = b_2 \\ \vdots \quad\ & \ \ \vdots \qquad\qquad\ \ \vdots \qquad \vdots \\ a_{m1}x_1 + \ & a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{aligned}$$

The example above is a system of (simultaneous) equations where there are M linear equations and N (economic) variables. Each M equation describes the interlink among each of the N variables in a ***linear*** fashion; the model is then called ***linear system***. Note that the model can be set up and take into account some features of ***non-linearity***, in which each equation can be expressed in terms of ***non-linear form***. That class of model will then label as the ***non-linear system***.

To understand the common structure and basic terminologies (jargons) used in mathematical economics, consider "*Example 1.B*" which is the reformulation of the basic market equilibrium model using mathematical expression.

Example 1.B: Consider the market equilibrium model as an example.

Equation 1: $Q_d = a - bP - cY$;

Equation 2: $Q_s = d + eP - fT$;

Equation 3: $Q_d = Q_s$

Notations defined as follows:

- Q_d = quantity demanded, Q_s = quantity supplied, P = price,
- Y = income, T= temperature ➔ variables
- (a, b, c, d, e, f) = *coefficients* or *parameters* that are all positive

Form the given model, there are *three equations*. There are *a number of variables*. To organize our mathematical thinking in a systematic fashion, let's consider the following issues.

1) **Treatment of the type of equations:**

   - Behavioral equations

   - Equilibrium conditions

2) **Treatment of the variables:**

   - Single equation approach:
     - Independent variable (*Explanatory*)
     - Dependent variable

   - System-based approach

     - **<u>Exo</u>**genous variables = treated as "***given***" in the model. Their values are given as known thing, and *will not be solved* within the model.
     - **<u>Endo</u>**genous variable = set of variables that will be solved for, determined with in the model.

1.2.1) Static Equilibrium analysis in Math framework!

From the example 1.B, the most common goal of studying the model is to study the *equilibrium analysis*, i.e. to solve for the solution of the simultaneous equations. As the exogenous variables are treated as given, solution to the system of equation would represent the *endogenous equilibrium solution*. Technically speaking, endogenous equilibrium solution is mathematically called the *reduced-form equation*.

Reduced-form equation is the mathematical function that describes the behavior of the endogenous equilibrium solution. *Typically, the function would depend on (i) exogenous variables and (ii) parameters.*

Example 1.C: National income model

A very basic/simple version of Keynesian cross model

- Consumption function: $C = \mathrm{C}(\mathrm{Y_d}; \mathrm{a}, \mathrm{b})$
- Disposable income: $\mathrm{Y_d} \equiv Y - T$
- Tax equation: $T = T_0$
- Investment equation: $I = I_0$
- Government equation: $G = G_0$
- Equilibrium condition: $Y = C + I + G$

Variable Lists:

Y = income, C = consumption, $\mathrm{Y_d}$ =Disposable income, T = tax
I = investment and G = government expenditure

- ***Static equilibrium analysis***: characterizing the equilibrium solution of the model by solving for a solution of the system of equations. By the definition, the reduced-form solution takes the following *general-form representation*.

$$Y^* = f(I_0, G_0, T_0, a, b)$$

$$C^* = g(I_0, G_0, T_0, a, b)$$

$$Y_d^* = f(I_0, G_0, T_0, a, b)$$

- If we assume the function form, the reduced-from solution would have an analytical (specific) solution which is specific to the assumption on the functional form of behavioral equation.
    - For example, we might assume that $\mathrm{C}(\mathrm{Y_d}; \mathrm{a}, \mathrm{b}) = \mathrm{a} + \mathrm{bY_d}$

Keywords: *Endogenous equilibrium solution, Reduced-form equations, general-form solution, specific-form solution*

1.2.2) Comparative Static analysis in Math framework

Having solved for the equilibrium solution, what economists usually ask is what would happen to the equilibrium if something, previously assumed to be fixed, has changed.

Example 1.C (cont.): National income model

- From the example 1.B, it is straightforward to solve for all the endogenous equilibrium solutions, Y^*, C^*, Y_d^*.

A very basic/simple version of Keynesian cross model
- Consumption function: $C = \mathrm{C}(\mathrm{Y_d}; \mathrm{a}, \mathrm{b})$ → $c = a + bY_d$
- Disposable income: $\mathrm{Y_d} \equiv Y - T$
- Tax equation: $T = T_0$
- Investment equation: $I = I_0$
- Government equation: $G = G_0$
- Equilibrium condition: $Y = C + I + G$

1) plug all Equations into the Equilibrium condition

2) ⇒ $Y = a + bY_d + I_0 + G_0$
$= a + b(Y - T_0) + I_0 + G_0$
$= bY + a - bT_0 + I_0 + G_0 \Rightarrow Y = \dfrac{a - bT_0 + I_0 + G_0}{1-b}$

- Numerically, if a = 1, T_0 = \$0 I_0 = \$1, G_0 = \$1 and b = 0.5, this yields us,

$$Y^* = \frac{a - bT_0 + I_0 + G_0}{1-b}$$

$$= \frac{1 - 0.5(0) + 1 + 1}{1 - 0.5} = \frac{3}{0.5}$$

$$= 6$$

Question: What if G is now changed to \$2, how big is the change in Y* and C*?

Answer:

New Y* = $\frac{1-0.5(0)+1+2}{1-0.5} \rightarrow \frac{4}{0.5} = 8$

A \$1 increase in G causes an increase in Y* by ___________

old G — new G

1 — 2 $\rightarrow \Delta G = 1$

old y^* — new y^*

6 — 8 $\rightarrow \Delta y^* = 2$

In the later part, we will try to figure out the value of *multiplier* when we don't assign any numerical values of exogenous variables.

- The idea is simple. *We just apply the derivative method to the equilibrium solution function.*
- That is, we calculate the value of $\frac{\partial Y^*}{\partial I_0}, \frac{\partial Y^*}{\partial G_0}, \frac{\partial Y^*}{\partial a}, \frac{\partial Y^*}{\partial b}$.

$$y^* = \frac{a - bT_0 + I_0 + G_0}{1-b}$$

Chapter 2: Sets, relation and function

2.1 Sets, relation and function in economics

- Relation
  - **Set:** A collection of items/objects. Each item or object in the set must share some properties/characteristic to the group/set where it belongs.
  - Element of one set (sets) can be associated with element of the other set, through a **relation**.
    - Mathematically, a relation represents a prescribed "**mapping rule**" between element in one set and element of the other set, i.e. set-to-set mapping.
    - Notionally, we write R:A→B if the relation describes the link/association between "set A" and "set B".
  - Types of relation
    - One-to-one
    - Many-to-one
    - One-to-many

- Function is two special classes of relation
  - One-to-one
  - Many-to-one

Domain and Range

To properly define any mathematical functions, the function must be associated with domain/range. See example 2.A

Example 2.A : $y = f(x) = \sqrt{x} - 3$

Domain for x $x \geq 0$

Range for y $y \geq -3$

<u>General function *and* Economic function:</u>

To present economic relationship in math form, we need to be very careful about the restricted sets of domains and ranges that makes the function/relation *economically meaningful*. See example 2.B below!

Example 2.B: Does the domain set for X continue to be the same as before if we now are using X to represent price, and y to represent quantity? Discuss, why or why not!

$$y = f(x) = \sqrt{x} - 3$$

quantity   Price

① Price must be non-negative number. ($x \geq 0$ is okay)

② what if $x = 4$; what is the corresponding value of y. Make sense

An alternative characterization is to use the step function →

$$y = \begin{cases} 0 & ; \ 0 \leq x \leq 9 \\ \sqrt{x} - 3 & ; \ x > 9 \end{cases}$$

2.2 Types of function commonly used in Economics

2.2.1) Linear function

A linear function takes the form:

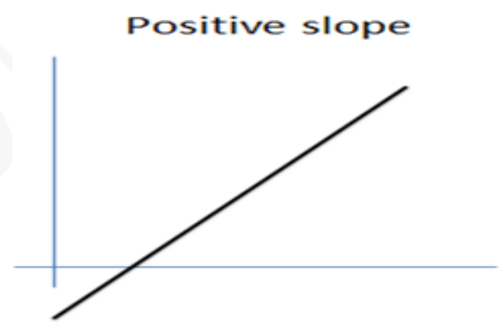
$$y = a + mx$$

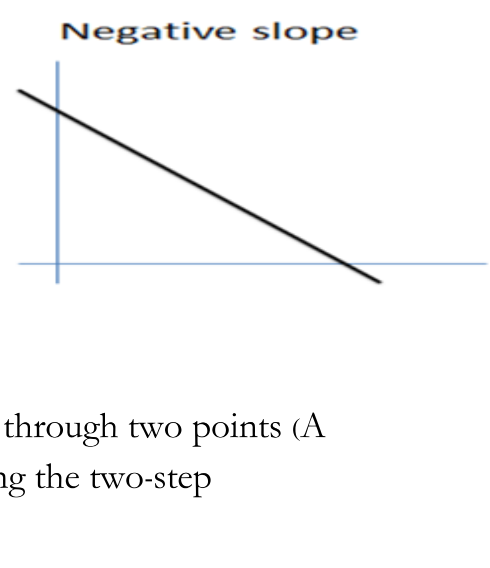
$$a = intercept\ \ on\ y - axis\ (x = 0,\ y = a)$$

$$m = slope$$

What is the slope?

- A change in "y" with respect to a change in "x".
- Standard notation refers to $\frac{\Delta y}{\Delta x}$ as the slope of a function.
- Generally, the value of slope can be varied with respect to changing values of X.
- But for the linear function, it's not. $\frac{\Delta y}{\Delta x} = m$ ➔ constant!
- Graphical illustration of a linear function depends on the value of
  - m > 0; positive slope = upward-sloping curve
  - m < 0; negative slope = downward-sloping curve





Note: With the constant slope, suppose we know a line goes through two points (A and B), we can derive the equation representing that line, using the two-step procedure.

- Steps 1: Deriving slope
- Steps 2: Solve for "a" by using slope and one of the two points given.

See **example 2.C** below

Example 2.C: Suppose a research has a priori belief that the relationship between "q" and "p" is linear. From the survey, the research only knows that when p is equal to 1, q is equal to 4. Meanwhile, when p is equal to 5, q is then equal to 0. Find the equation of the line that represents the p-q relationship? What does the relationship represent?

$(p,q) \Rightarrow A(1,4)$: initial

$B(5,0)$: old

Step 1

$$\text{slope } (m) = \frac{0-4}{5-1} = -\frac{4}{3}$$

Step 2

$$Q = a + mp$$

$$= a - \frac{4}{3}p$$

B(5,0) makes the equation satisfied.

$$0 = a - \frac{4}{3}(5) = a - \frac{20}{3}$$

$$\therefore a = \frac{20}{3}$$

$$Q = \frac{20}{3} - \frac{4}{3}p$$

Example 2.E: Suppose $Q_x = 10 - P_x + bP_y$ where

Q_x is quantity demanded for x.

P_x is price of good x.

P_y is price of good y.

What could be summarized as the relationship between good x and good y under which "b" is positive, zero, and negative?

Conclusion

Given $Q_x = k - aP_x + \boldsymbol{b}P_y$, we can summarize that if

- b = positive ➔ an increase in P_y ➔ Q_x increases ➔ *substitute product*
    - I-phone v.s. Samsung
- b = negative ➔ an increase in P_y ➔ Q_x decrease ➔ *complement product*
    - Coffee *and* Tea
- b = 0 ➔ an increase in P_y ➔ Q_x stay the same ➔ *neither substitute nor complement*
    - Toilet paper and Ferrari

Slope v.s. elasticity

General notation:

Consider two points A0(x0,y0) and A1(x1,y1). Let ε_x^y be the elasticity of "y" with respect to "x". The *"point"* elasticity is then defined as follow

$$\frac{\%\Delta y}{\%\Delta x} \equiv \frac{\frac{y_1 - y_0}{y_0}}{\frac{x_1 - x_0}{x_0}} = \frac{y_1 - y_0}{x_1 - x_0} * \frac{x_0}{y_0} = \frac{\Delta y}{\Delta x} * \frac{x_0}{y_0} = Slope * \frac{x_0}{y_0}$$

Notice that the value of elasticity *depends* on the *starting (initial) value*, i.e. (x0,y0).

An alternative version to the "point" elasticity is called "*arc*" elasticity. The definition is given by,

$$\frac{\%\Delta y}{\%\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} * \frac{\frac{x_0 + x_1}{2}}{\frac{y_0 + y_1}{2}} = \frac{\Delta y}{\Delta x} * \frac{x_0 + x_1}{y_0 + y_1} = Slope * \frac{x_0 + x_1}{y_0 + y_1}$$

Example 2.F: *Elasticity of demand*

Suppose we know any two points on the demand curve, namely A$(Q_0 = 4, P_0 = 1)$ and B$(Q_1 = 1, P_1 = 5)$ where A is the initial point and B is the terminal point

Elasticity of demand of demand is then defined as

$$\frac{\%\Delta Q}{\%\Delta P} = \frac{Q_1 - Q_0}{P_1 - P_0} * \frac{P_0}{Q_0} = \frac{\Delta Q}{\Delta P} * \frac{P_0}{Q_0}$$

More about the elasticity of demand

- Own-price elasticity of demand is always less than or equal to zero! (why?)
  - Negative sign only implies that negative relationship between price and quantity demand.
  - Absolute number of the figure tells us about the degree of sensitivity of quantity demanded to the price change.

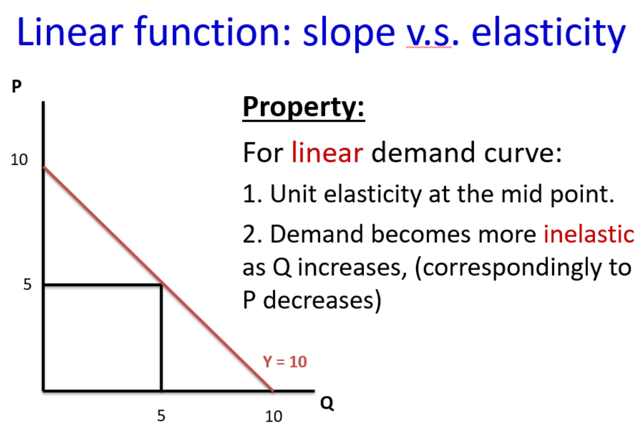
- Conventionally, we classify the case for elasticity of demand into three cases.

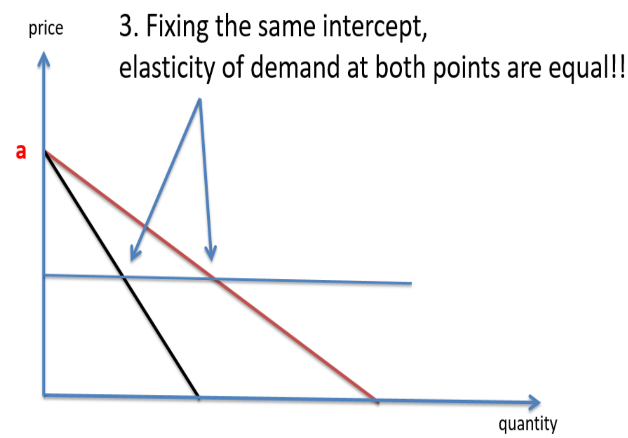
$$\left|\frac{\%\Delta Q}{\%\Delta P}\right| : \begin{array}{lll} > 1 & ; & elastic \\ = 1 & ; & unit\ elastic \\ < 1 & ; & inelastic \end{array}$$

- Two important implications of elasticity for practical use.
  - Boosting revenue; when to raise the price!
  - Optimal commodity taxation; DW loss proportional to elasticity

Example 2.G: Elasticity of linear demand curve and some important properties

Suppose that $p = 10 - Q$, derive the formula for the elasticity of demand





Exercise 2.A:

2.A.1) Given a demand function by $p = a - bQ$, derive the formula for the elasticity of demand, and show that the third property holds

2.A.2) Given the market supply $p = c + dQ$ where $d \geq 0$, show that

(i) elasticity of supply is always greater than 1 if $c > 0$,

(ii) elasticity of supply is always eqaul to 1 if $c = 0$,

(iii) elasticity of supply is always less to 1 if $c < 0$.

2.2.2) Non-linear functions

While the linear functional form is useful, there are several drawbacks pertaining to its application. One of the several drawbacks is about its shape of function which implies a constant slope.

- **Example:**
  - For simplicity, Keynesian assumption typically assumes that aggregate consumption is linearly related to the level of income, i.e. $C(t) = a + bY(t)$.
  - Does it make sense to assume that each individual household has a linear consumption function?
  - Perhaps not! SES Panel survey shows that the rich tend to consume less out of each additional unit of income they have earned. Thus, MPC should be "decreasing" as income increases.
  - Given the argument above, neither the slope of consumption function should be constant nor the form of consumption function could be truly represented by a linear equation. (In fact, the consumption function should be a *concave function*!)

Given the motivated example, many economic functions are represented in terms of non-linear relationship. Mathematically, there are two broad classes of non-linear functions that economists have commonly used.

- Polynomial function
- Exponential/Logarithm function

2.2.2a) Polynomial function

Polynomial takes the form

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N$$

N = 1 ➔ linear ➔ straight line

N = 2 ➔ Quadratic function ➔ Parabola shape (U-shape)

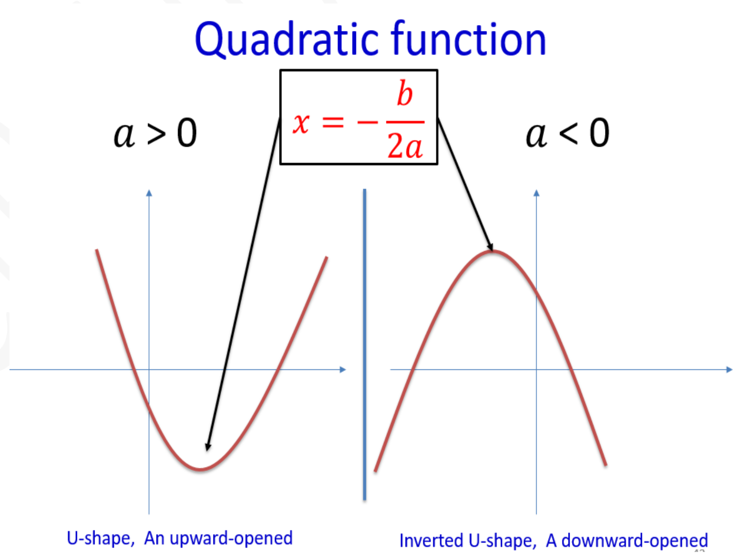
N = 3 ➔ Cubic function ➔ Uncertain. Depends, requires calculus.

The Quadratic function

Consider the polynomial function with degree 2.

- Let's write the function in the form that you are used to.

$$y = ax^2 + bx + c$$



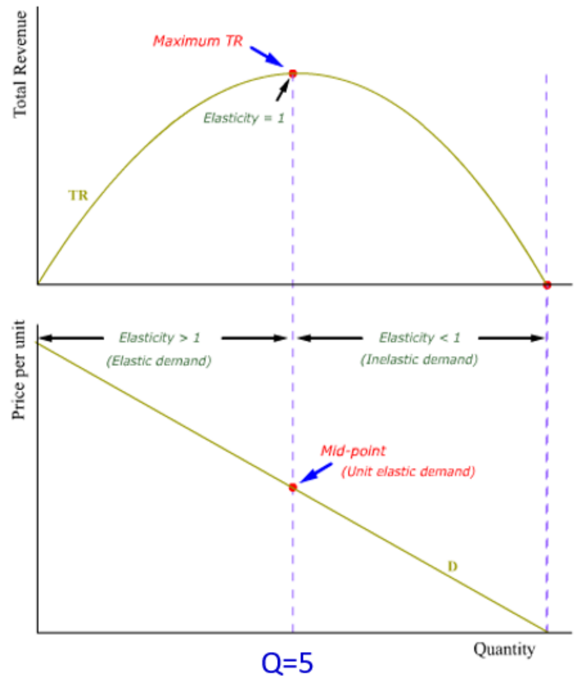
Example 2.I: A monopolist firm faces the market demand given by $P = 10 - Q$. Consider the following questions if the cost function $C(Q) = 4Q$.

- What is the revenue-maximizing level of output?

- What is the break-even output?

- What is the profit-maximizing level of output?

Quadratic function: revenue function/break even analysis



$$\left|\frac{\%\Delta Q}{\%\Delta P}\right| : \begin{array}{lcc} > 1 & ; & elastic \\ = 1 & ; & unit\ elastic \\ < 1 & ; & inelastic \end{array}$$

Revenue maximizing output:
unit elasticity

Profit-maximizing output:
under region with **elastic** demand

48

Exrcise 2B. Consider a function that relates tax revenues R, in billions of dollars, to the average tax rate t such that $R = 350t - 500t^2$.

(a) What tax rate(s) is consistent with raising tax revenues equal to $60 billion?

(b) What tax rate(s) is consistent with raising tax revenues equal to $61.25 billion?

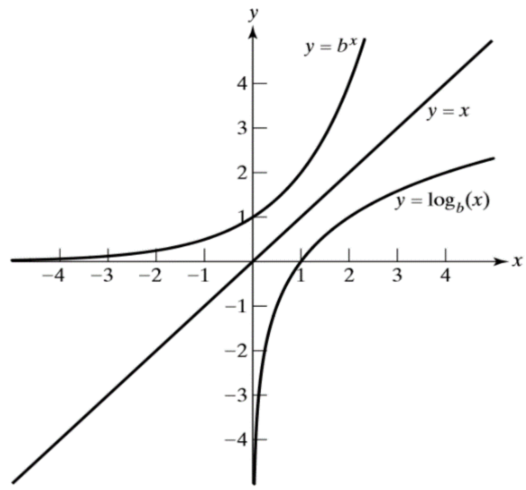
(c) What tax rate is consistent with the maximum tax revenue?

2.2.2b) Exponential and Logarithm

Two other non-linear functions commonly used in economics analysis are *exponential* and *logarithm*. The two functions are inverse function of each other.

Exponential : $y = b^x$

Logarithm : $y = \log_b x$



Natural number: "e" → $y = e^x$ or $y = \ln x$

Rule of exponential:

Expoential transformation of a product,
For any two positive varaiables x and y,

$$b^x * b^y = b^{x+y}$$

Expoential transformation of a Quotient, For any two positive varaiables x and y,

$$\frac{b^x}{b^y} = b^{x-y}$$

Expoential transformation of a Power, For any two positive varaiables x and y,

$$(b^x)^y = b^{xy}$$

Rule of logarithm

Logarithmic Transformation of a Product For any two positive variables X and Y,

$$\log_b(XY) = \log_b(X) + \log_b(Y).$$

Logarithmic Transformation of a Quotient For any two positive variables X and Y,

$$\log_b\left(\frac{X}{Y}\right) = \log_b(X) - \log_b(Y).$$

Logarithmic Transformation of an Exponent For any positive variable X and any values of λ,

$$\log_b(X^\lambda) = \lambda \log_b(X).$$

Common application of exponential/logarithm function

1. **Growth calculation**

$$X_t = (1+g)X_{t-1} \quad \rightarrow \quad X_t = (1+g)^t X_0$$

"g" is net rate of growth ; "$1+g$" is gross rate of growth

2. **Financial mathematics**
   a. Compound interest rate:
      i. $X_{t+n} = (1+r)^n X_t$
      ii. $X_{t+n} = \left(\left(1+\frac{r}{k}\right)^k\right)^n X_t = \left(1+\frac{r}{k}\right)^{nk} X_t$
      iii. $X_{t+n} = e^{rn} X_t$
   b. Present value:
      i. $X_t = X_{t+n}\left(1+\frac{r}{k}\right)^{-nk}$
   c. Asset valuation:
      i. Asset → pays you streams of cash flow
      ii. Value of asset = Value of cash flow; sum of present value of cash flow at each point of time.

Example 2.J: Bond pricing formula

A Two-year bond offers a stream of payment for \$10 in every single year until it is matured. At the maturity date, this bond also delivers the principal amount at \$100. If market interest rate is r% per annual, what would be the fair price of the two-year bond?

Chapter 3: Equilibrium and Comparative static analysis: Linear model

3.1 Linear economics model

What we have been discussing so far centers on (i) how to characterize relationship of economic variables using mathematical function/equation, and how to (ii) properly define the function/equation in economics analysis.

What we are going next is to

- Write down some basic linear economic models (in mathematical form)
  - Set of system of equations.
  - Interlink among variables in the model.
- Solve for the solution of the model => *Static analysis*
  - (Simultaneous) Solution to the system of equations
- Analyze behavior of the endogenous equilibrium solutions => *Comparative static analysis*

List of the economic models that we will be taking about include the follows.

- Micro-market equilibrium model
  - Single market equilibrium (Partial analysis)
  - Multi-market equilibrium (General analysis)
- Macroeconomic model
  - Keynesian cross
  - IS-LM model

Before we proceed, let's review some of the basic tools, i.e. *solving for the solution of system of equation.*

3.2 System of equations

Sytem of equation is a mathematical formulation, comprising of *N-unknown* variables **and** *M-equations.* The equations could be *linear, non-linear, or a mixture of both.* The objective of working with system of equation is to solve for a set of N-unknown variables that simultaneously satisfies all the M equations; that is, we derive for the solution of the system.

For now, we will only focus on 2-by-2 or 3-by-3, and restrict our attention to the case in which system of equations only comprises of linear equations. Under these two cases, the system is solvable by *pencil and paper.* For any larger scale of the system of equations, we might have to resort on the matrix algebra to solve for the solution. This topic will be discussed in the subsequent chaper.

Example 3.A: Solving for the solution of the system of equations

$X + Y = 7$ — 1

$3X - 6y = -15;$ — 2

$3 \times 1 \Rightarrow 3x + 3y = 21$ — 3

$3 - 2 \Rightarrow (3x + 3y) - (3x - 6y) = 7 - (-15)$

$3x + 3y - 3x + 6y = 22$

$9y = 22 \rightarrow y = \frac{22}{9}$

since $x + y = 7 \rightarrow x = 7 - y = 7 - \frac{22}{9} = \frac{41}{9}$

Example 3.B: Distinguishing the type of solution

- System 1: $2x + 3y = 7$ and $x - y = 5$.
- System 2: $y = -2x + 6$ and $4x + 2y = 12$.
- System 3: $x + 2y = 14$ and $3x + 6y = 8$

3.3 Market Equilibrium model

The market equilibrium model is a mathematical formulation of basic underlying equilbirum analysis framework in economics. In the model, we ask two simple questions. First, at what price, people trade? And, seconld, how much do they trate? The two jargons commonly shown up in the economic readings are "*market-clearing price*" and *market-clearing quantities*". The objective of equilbirum model is to predict the two objects in question.

When one anlyze the equilbirum concepts, there are two distinct notions, namely, partial equilibrium (PE) vs general equilibrium (GE). The former considers the equilibrium outcome of a particular market, while treating the rest as given. The latter considers the feedback interaction across markets. Usually, simple analysis will start from PE approach as it is easier, and require less computational burdern. However, this presumption is made to simplify the analysis. In some cases, it might be resonally working well to assume the given assumption. For example, if we want to analyze the equiblirum determination of Thai GDP, we might consider US GDP as given. However, to analyze the equilibrium determination of China GDP, it might not make sense to assume that US GDP can be given; they are both equally important to each other.

<u>**Example 3.C**</u> Compare and contrast between PE and GE model

Partial

$$Q_d^x = a - bP_x + cP_y$$
$$Q_s^x = d + eP_x$$

P_y are given exogenously

General

$$Q_d^x = a - bP_x + cP_y$$
$$Q_s^x = d + eP_x$$

$$Q_d^y = f - gP_y + hP_x$$
$$Q_s^y = j + kP_y$$

(P_x, P_y) are solved for together.

3.3.1) Partial equilibrium model

Example 3.D Suppose that *market* demand and supply equation can be given by the following two equations:

$$Q_d = 30 - 5Y - 3P$$

and $$P = 2 + Q_s$$

Find the market equilibrium for quantity of output and price.

Caution: Equilibrium notion requires the information of the *market* demand and *market* supply. Mostly, this information needs to be derived from the given individual demand and supply.

Definition:

1. For a given level of price, market demand is the total quantity demanded of each individual (consumer/buyer) combined.

$$Q^d = \sum_{i=1}^{N} Q_i^d$$

where Q^d is total quantity demanded at the market level.
Q_i^d is the quantity demanded for i-th individual.

2. For a given level of price, market supply is the total quantity supplied of each individual (firm/seller) combined.

$$Q^S = \sum_{i=1}^{M} Q_i^S$$

where Q^S is total quantity demanded at the market level.
Q_i^S is the quantity demanded for i-th firm.

Two methods for the aggregation!

- Direction summation approach
  - Apply basic sigma operation
- Horizontal summation approach
  - Deriving market demand curve from individual demand curves
  - Find the mathematical equation that represents the derived market curve

See Example 3.E below

Example 3.E: Market demand

Consumer 1: $Q_1^d = 3 - P$

Consumer 2: $Q_2^d = 2 - P$

Direct Summation:

Deriving market demand curve from individual demand curves.

What is the mathematical equation that represents the market demand curve when using the horizontal summation method? Derive the equation/function.

Example 3.F: (market supply):

A market has two firms, A and B. Each has the supply equation given by,

Firm A: $P = 15 + Q_a$

Firm B: $Q_b = 0.5 \cdot P - 5$

Derive the market supply equation

Example 3.G: Solve for the market equilibrium using the information in **Example 3.E** and **Example 3.F.** Justify your answer!

Some extensions: Government interventions

Taxation

- The first type of government intervention is *taxation*
- Micro-market model allows us to understand the impact of commodity tax.
- Basically, there are two types of commodity tax.
    - **Unit tax** ➔ \$t per each unit of production.
    - **Ad-valorem tax** ➔ t% of value.
- Let's start from the *impact of unit tax*. Then, we will come back to the analysis on *Ad-valorem tax* later.
- The purpose of this part is to show:
    - A rigorous approach to analyze the impact taxation on price and allocation in the market equilibrium.
    - Showing that *the implication of tax on the welfare*, and *distribution of welfare cost* among stakeholders in the market.
        - Tax burden
        - Deadweight loss

Impact of the unit taxation

First, recall that the basic set up of linear market model typically takes the form:

Demand: $p \quad = a - bQ^d \quad ; \ a \geq 0, \quad b \geq 0.$

Supply : $p \quad = c + dQ^s \quad ; \ d \geq 0.$

- Equilibrium can be then solved by imposing the market clearing condition, i.e. $Q^d = Q^s$

When the government imposes tax, the followings happen.

1. With the imposition of tax, *price that consumer's paying will no longer be equal to price that producer's receiving.*
    - The single "p" concept that underlies the basic set up of linear market model would no longer be sensible!
    - So, to analyze the impact of taxation, we need to make it clear from now that
        - p^s = price that producer gets
        - p^d = price that consumer pays
        - $p^s \neq p^d$ ➔ price gap *or* price wedges

Behavior of buyers and suppliers would focus at their ***perceived prices***.

That is, the market model needs to be rewritten in the following way.

Demand: $p^d = a - bQ^d \quad ; \ a \geq 0, \quad b \geq 0.$

Supply : $p^s = c + dQ^s \quad ; \ d \geq 0.$

2. With the imposition of tax, equilibrium allocation would be affected!

   - *VERY SIMPLE,* we solve for equilibrium by imposing that $Q^d = Q^s$.
   - BUT!!!!

Example 3.H Can we solve for the equilibrium by simply imposing the only market clearing condition?

Solvable? If not, what else do we need?

Example 3.I Suppose that demand/supply equation can be given by

$$P^d = 80 - 3Q^d$$

$$P^s = 10 + 2Q^s$$

- Find the market equilibrium.

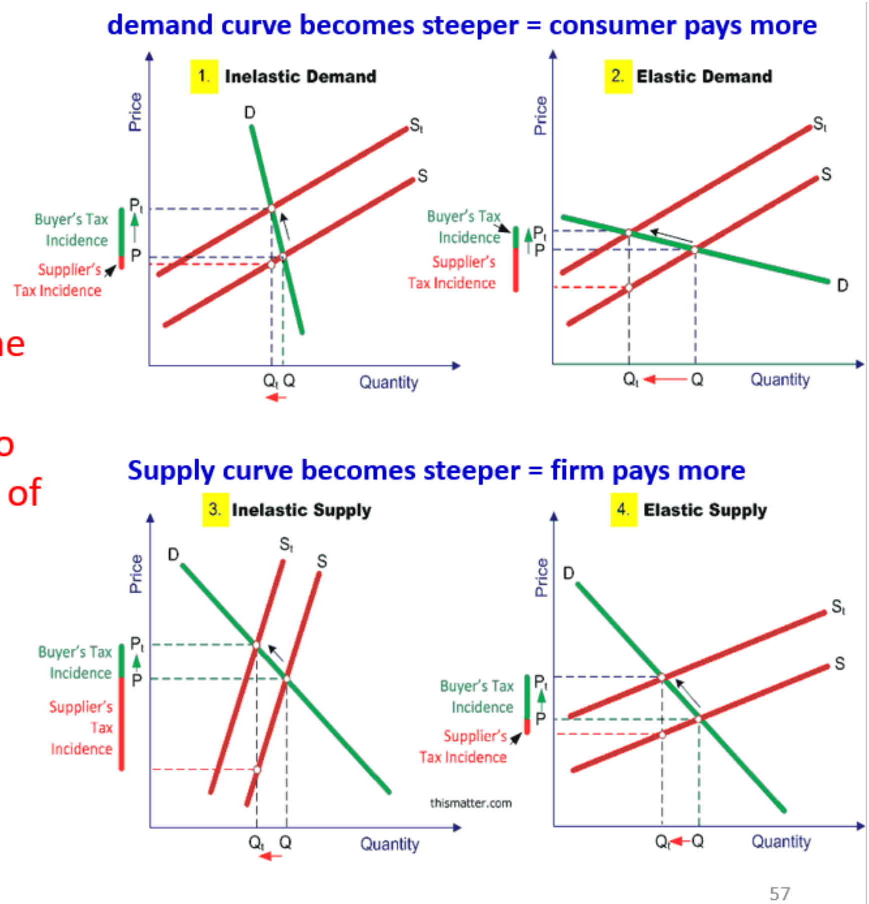
- Suppose that government imposes tax on producer equal to \$5 per unit, determine market equilibrium under taxation.

- How much is the tax revenue that the government can collect in the equilibrium?

- Calculate the tax burden.

Excess burden/Tax incidence:

Main Idea: Whoever cannot adjust themselves instantaneously to the new situation would unfortunately have to bear a larger portion of the tax burden.



Example 3.J: Excess burden *formula under linear model & Tax-Revenue-maxizing tax rate*

Demand: $p^d = a - bQ^d \quad ; \ a \geq 0, \quad b \leq 0.$

Supply : $p^s = c + dQ^s \quad ; \ d \geq 0.$

- Solve for quantity and prices equilibrium when the unit tax is imposed. Analyze the result

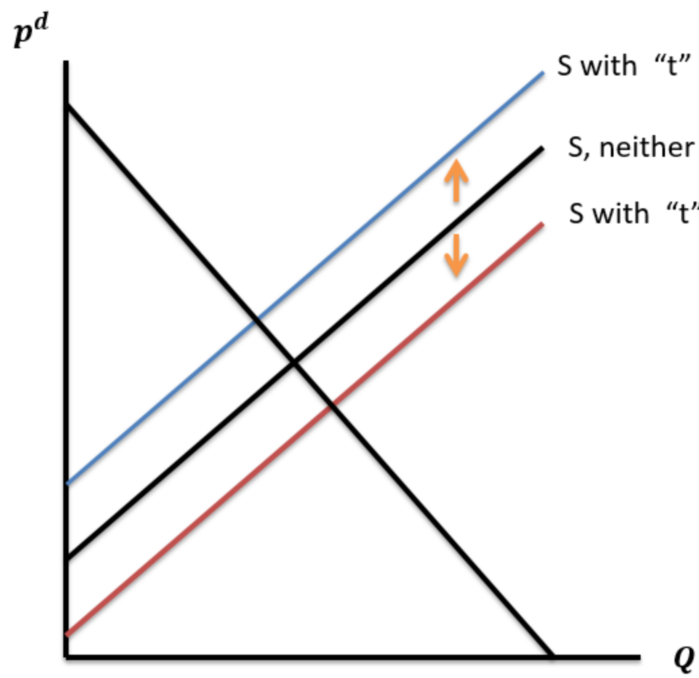
- Derive the excess burden formula for buyers and sellers

- Calculate the tax rate that maximizes the tax revenue of government.

Unit-subsidy?

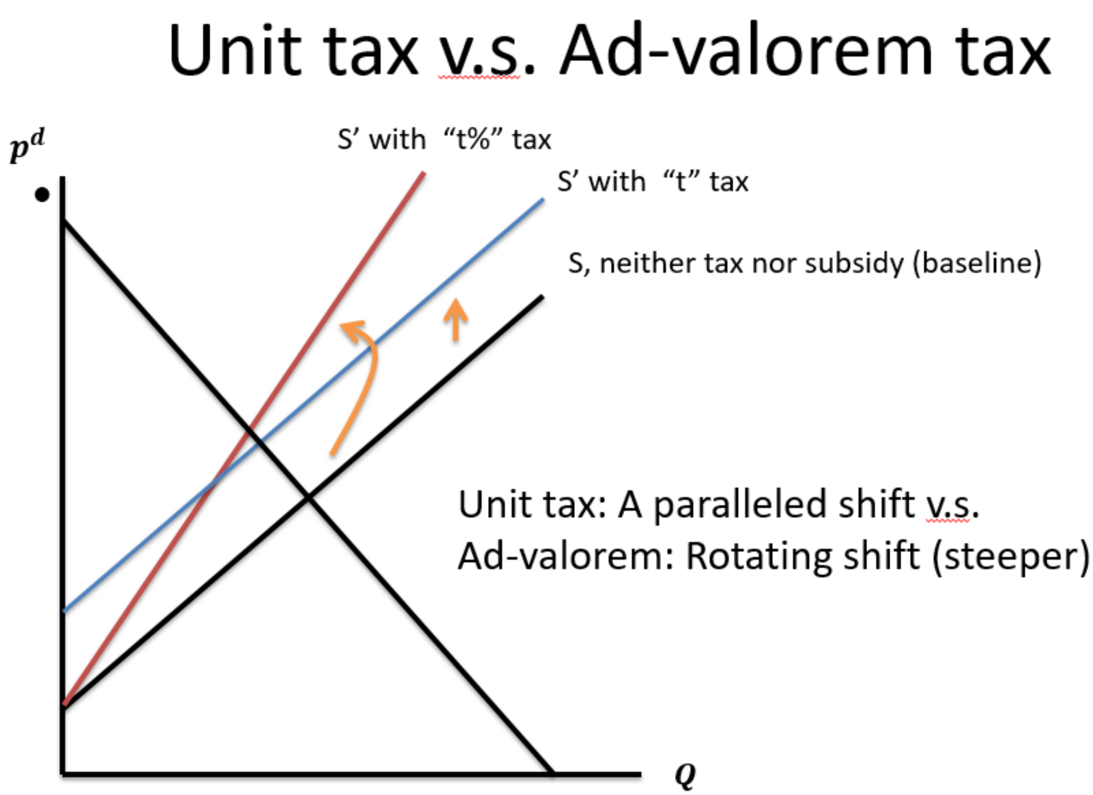
- Do we have to start everything from the beginning to derive the implications of subsidy program for market equilibrium?
    - *No*, the analytical result for the impact of subsidy would have been the same, because we can treat subsidy as ***a negative tax***. *"t" is negative number!*
    - The burden concept would be changed into the benefit received, i.e. who's getting more benefit.

Tax v.s. Subsidy (unit-based)



Ad-valorem tax

- Tax by t% of price[1]. An example is the value-added tax.
- *Most* Ad-valorem taxes are imposed at the point of purchase, i.e. imposing on consumers
- If the tax is imposed on consumer, it would be that: $p^d = p^s\left(1 + \frac{t}{100}\right)$.



<u>Example 3.K</u> ***Price control and Welfare***

Consider the market for apartment rentals in Chicago. The price of rent is determined by the following system of equations.

Demand: $p = -2q_d + 160$

Supply: $p = q_s + 10$

- What is the equilibrium price and quantity in the market for apartment rentals?

- Suppose the government tries to control the rent prices through a price ceiling of $40. Discuss the implication of this policy. Is there any deadweight loss?

3.3.2) Multi-market model (*General equilibrium model*)

So far, we have only analyzed the market equilibrium concept when we assume that there is only single market in the economy. In fact, the economy comprises of several markets, each of which is interdependence. The notion of equilibrium concept that we use so far must be extended to fit with the more realistic environment where we have N markets. To understand this, consider the example below.

<u>Example 3.K</u> Let demand and supply of market for goods A and B be

$$D_a = 20 - 3Pa - 2Pb$$

$$S_a = 12 + 2Pa + 5Pb$$

$$D_b = 4 - Pa - 3Pb$$

$$S_b = -1 + 2Pa + 4Pb$$

where D_i and S_i represent the demand and supply for the market of good "j",

P_i is the price of goods j.

- What do we see about the nature of demand and supply in each market? Are they independent to each other?

- What is the appropriate notion of the equilibrium?

- Find the equilibrium price and quantity of the market for goods A and the equilibrium price and quantity of the market for goods B.

3.4 Simple macroeconomics models

We will revisit two important classes of the macroeconomic model, previously discussed in EE212. The two models are Keynesian cross model (DAE model), and the IS-LM model.

- **The Keynesian cross model is an example of single market equilibrium model.**
  - Analyze equilibrium determination of "output" in the "goods market".
    - Output = real GDP = National income
    - Goods market: market where outputs are sold, and purchased by economic agents.
  - Treating financial sector as given, i.e treating market interest rate as given!
    - Making sense under the *liquidity trap* environment
  - Abstract from production decision of firms
    - Those who studied EE212 might have question about the role of "price"
    - In this model, firm will be passive, i.e. producing as much as economic agents want.
  - The model provides a good foundation for understanding the implication of *fiscal policy*.

- **The IS-LM model is an example of Multi-market equilibrium model.**
  - Endogenize the market interest rate.
  - Introduce the *feedback/interactive effect* between financial market and goods market.
  - Yet, we retain the assumption where firms are passive. (Price is fixed)
  - The model provides a good foundation for understanding the implication of *fiscal and monetary policy*.

Where we are headed!

- Revisting these models, mathematically.
- Solving for equilibrium solution of endogenous variables
- Some comparative static analysis: *analyze behavior of endogenous variables.*
- Policy issues
  - *Multipliers; Crowding-out effects; Policy effectiveness*

3.4.1) The basic workhorse: the Keynesian cross model

From example 1.B, the simplest workhorse for the Keynesian cross model is the one that assumes a closed economy with no government sector. The model can be explained by 3 equations followed.

$Y = C + I$ ➔ equilibrium condition

$C = \mathrm{C(Y)} = \mathrm{a} + \mathrm{bY}$ ➔ behavior of consumption

$I \ = I_0$ ➔ exogenous investment (take it as given.)

- Exogenous: I_0.
- Endogenous: C and Y.
- Parameters: a (autonomous consumption), b (mpc)

Static analysis: Equilibrium solution of endogenous variables takes the form

$$\mathrm{Y}^* = \frac{\mathrm{a} + \mathrm{I}_0}{1 - \mathrm{b}}$$

$$\mathrm{C}^* \ = \mathrm{a} + \mathrm{b}\left(\frac{\mathrm{a} + \mathrm{I}_0}{1 - \mathrm{b}}\right)$$

Sensitivity analysis: suppose that $\mathrm{I}_0 \rightarrow \mathrm{I}_0^{'}$, what would happen to the equilibrium?

$$\Delta Y^* = Y^{*'} - Y^* = \frac{1}{1 - b} * (I_0' - I_0)$$

$$\Delta C^* = C^{*'} - C^* = \frac{b}{1-b} * (I_0' - I_0)$$

Extensions: *Simplicity and reality*

One can enrich the basic model in several directions; this allows us to yield the model that betters reflecting how the world/economy actually works. An exhaustive list of possible extension includes:

1. Endogenous investment
   - Investment also depends on level of income, i.e. I(y)
   - Income-accelerator framework, i.e. I(Δy) (*Dynamic*)

2. Government sector
   - Government spending
     - Typically treated as an *exogenous* variable.
     - Alternatively, G might be tied to the level of GDP.
     - Automatic stabilizer, i.e. counter-cyclical government spending rule.
   - Taxation
     - Consumption depends on disposable income, i.e. C(Y-T)
     - Income tax, i.e. T(y)

3. Effect of interest rate on private spending
   - Consumption and investment could also depend on interest rate, i.e. C(y,r) and I(y,r)
     - This extension reflects the idea that cost of borrowing matters to both households' and firms' spending.
   - Also, introducing the role of interest rate allows for the channel of linkage between real and financial sector!

4. Open economy
   - Equilibrium condition must include net export, i.e. X – M.
   - Need to model the behavior of "X" and "M".

Example 3.L: A full-version of closed-economy Keynesian cross model

Equilibrium (1)	$Y = C + I + G$
Consumption function (2)	$C = a + bY_d$
Disposable income (3)	$Y_d \equiv Y - T$
Tax rule (4)	$T = T_0 + t * Y$
Investment function (5)	$I = I_0 - I_1 r$
Government spending (6)	$G = G_0$

- Solve for the equilibrium income function?

$$y = c + I + G$$

$$= a + by_d + I_0 - I_1 r + G_0$$

$$= a + b(y - T) + I_0 - I_1 r + G_0$$

$$= a + b[y - (T_0 + t \cdot y)] + I_0 - I_1 r + G_0$$

$$= a + b[y(1-t) - T_0] + I_0 - I_1 r + G_0$$

$$= a + b(1-t) \cdot y - bT_0 + I_0 - I_1 r + G_0$$

$$y - b(1-t) \cdot y = a - bT_0 + I_0 - I_1 r + G_0$$

$$y^* = \frac{a - bT_0 + I_0 - I_1 r + G_0}{1 - b(1-t)}$$

- Multiplier of autonomous expenditures

3.4.2) The IS-LM model

- So far, we have seen a simple Keynesian cross model that *"r" is exogenously given.*
  - An increase in "r" reduces aggregate expenditure, and hence the level of income in the equilibrium.
- Is the assumption that "r" an exogenous variable sensible? Not *really*!..
- The IS-LM model is an extension of the Keynesian cross where
  - The model *endogenizes* the determination of *"real" interest rate* by modeling how interest rate is determined in the equilibrium.
- Conceptually, interest rate is determined from the equilibrium in money market, *where "real" money demand is equal to "real" money supply.*
- In economics, we typically assume that money demand depends on (i) income and (ii) interest rate.
- Since money demand also depends on income, equilibrium level of interest rate in the money market also depends on the level of income.
- As a result, income and interest rate will have to be simultaneously determining each other in the general equilibrium.

Money market modeling

Equilibrium condition	$L^d = \frac{M^s}{P}$
Money demand equation	$L^d = \mathrm{L}_0 + \mathrm{L}_1\mathrm{Y} - \mathrm{L}_2\mathrm{r}$
Money supply	$M^s = M_0^s$
Price level	P is given

Characterizing equilibrium in the money market

- Putting everything into the equilibrium condition

$$L_0 + L_1Y - L_2r = \frac{M_0^S}{P}$$

- Suppose that "Y" is given for now. Write "r" in terms of "Y" and "$\frac{M_0^S}{P}$", we yield **the LM equation**,

$$r = \frac{1}{L_2}\left(L_0 + L_1Y - \frac{M_0^S}{P}\right)$$

<u>Multi-market equilibrium in the IS-LM model</u>

The equilibrium solution for Y* and r* simultaneously satisfy the following two equations

IS equation: $Y = \frac{1}{1-b(1-t)}[a - bT_0 + I_0 - I_1r + G_0]$

LM equation: $r = \frac{1}{L_2}\left(L_0 + L_1Y - \frac{M_0^S}{P}\right)$

Example 3.M Consider the following model equations that explain the behavior of goods and money market.

$C = 0.8(Y - T)$
$T = 1{,}000$
$I = 800 - 20r$
$G = 1{,}000$
$Y = C + I + G$
$L^d = 0.4Y - 40r$
$M^S = 1{,}200$
$P = 1$

Derive the IS equation → Equilibrium in good markets
" Y v.s. r "

$$Y = C + I + G$$
$$= 0.8(Y - T) + 800 - 20.r + G$$
$$= 0.8Y - 0.8T + 800 - 20.r + G$$
$$Y - 0.8Y = -0.8T + 800 - 20.r + G$$
$$Y^* = \frac{-0.8T + 800 - 20.r + G}{0.2}$$
$$= 5(-0.8T + 800 - 20.r + G)$$

Derive the LM equation

← Equilibrium in money market
" r v.s. Y "

$$\frac{M^S}{P} = L^d$$

$$\frac{M^S}{P} = 0.4Y - 40.r$$

$$r = \frac{0.4Y - \frac{M^S}{P}}{40} = 0.01Y - \left(\frac{1}{40}\right) \cdot \frac{M^S}{P}$$

Solve for endogenous equilibrium solution of "Y", "C", "I" and "saving".

Conjoint the two markets "y" and "r" are simultaneously determined.

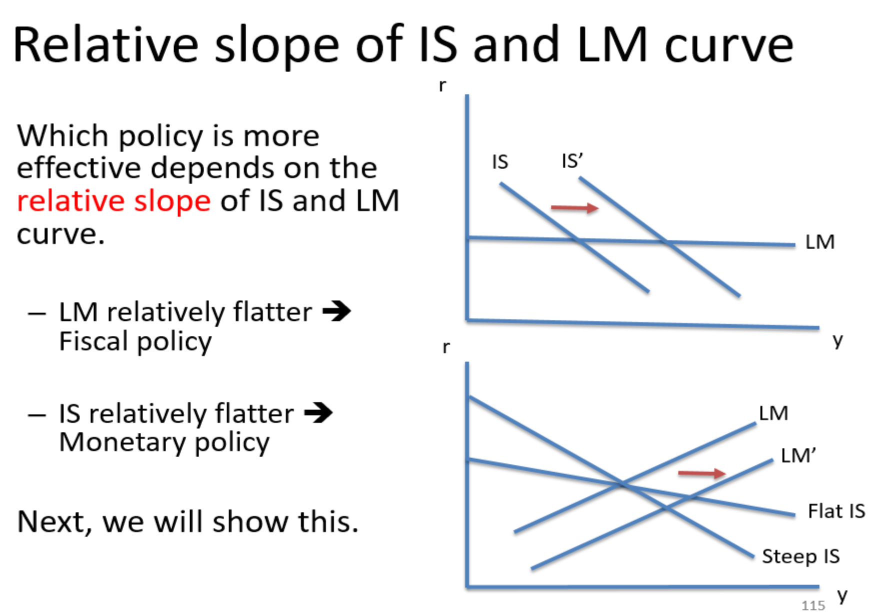
What happen to equilibrium "Y" when G increases by 200.

Back to when G is equal to 1000. What happen to equilibrium "Y" when money supply increases by 200.

Measuring in term of the change in output, which policy is more effective? Explain the reason.

The issue on Policy effectiveness!

- Effectiveness of a policy can be measured by the "*size of output change.*"
- To understand this in further detail, define $\frac{\Delta y}{\Delta Policy_i}$ as the multiplier of a *policy measure (variable)*, indexed by "i".
    - Policy: fiscal v.s. monetary policy
    - Policy *measures (or variables)*, e.g.
        - Fiscal *measures*: *increase/decrease tax or government purchase*
        - Monetary *measure*: *increase/decrease money supply*
- One can denote that "*policy-a*" is more effective than "*policy-b*" when the multiplier of "*policy-a*" is bigger than that of "*policy b*".
    - Consider for example, the multiplier of government purchases and tax.
    - In general, government purchase is more effective than tax policy as the multiplier of government purchase is bigger than that of tax.
- Comparing across the two types of policy, what we have seen from *EE212* are the following conclusions!
    - The flatter is the *IS curve*; the more effective is *monetary policy* in influencing output.
    - The flatter is the *LM curve*; the more effective is *fiscal policy* in influencing output.



Example 3.N: Using the IS-LM equation derived above, and show that

$$\frac{multiplier\ of\ M}{multiplier\ of\ G} = \frac{I_1}{L_2}$$

IS equation:

$$Y = \frac{1}{1 - b(1-t)}[a - bT_0 + I_0 - I_1 r + G_0]$$

LM equation:

$$r = \frac{1}{L_2}\left(L_0 + L_1 Y - \frac{M_0^S}{P}\right)$$

<u>**Economic Intuitions!**</u>

1. **The flatter is the IS curve**; *the more effective is monetary policy in influencing output.*

- To understand this, we discuss about how monetary policy works.
    - *Change in money supply affects the interest rate.* → *The change in interest rate affects private spending.* → *Change in private spending alters the output.*

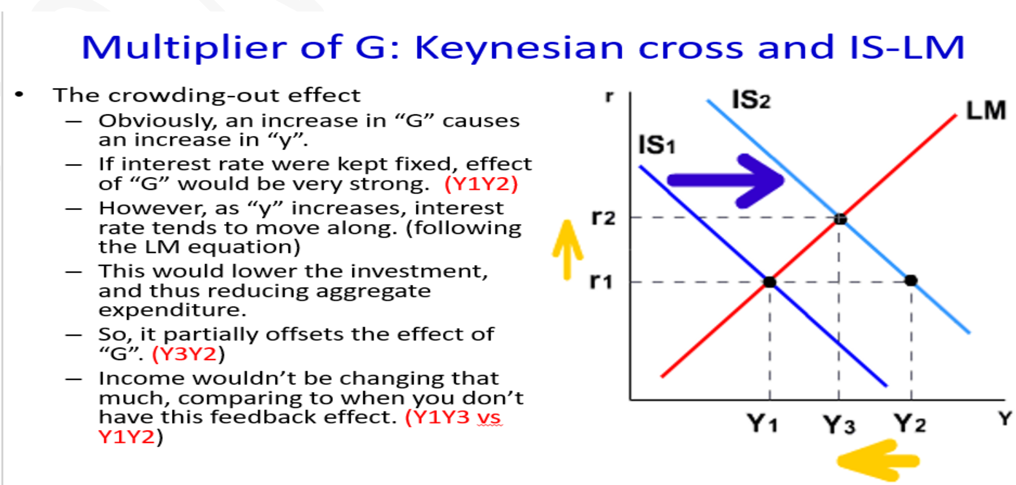
- How does this relate to the shape of IS curve:
    - When I_1 is high, the IS curve is flatter!
        - $Y = \frac{1}{1-b(1-t)}[a - bT_0 + I_0 - I_1 r + G_0]$
    - When I_1 is high, aggregate expenditure is highly sensitive to interest rate.
        - So, a small change in liquidity can generate large effect on private spending, and hence causing a large stimulative impact on total output.

2. **The flatter is the LM curve**; *the more effective is fiscal policy in influencing output.*

- To understand this, we discuss about how fiscal policy works.
  - *Suppose we consider the effect of government purchase.*
  - *The storytelling goes as follow: "Change in government purchase affects the private spending → Private spending affects output → Output affects interest rate, and partially reduce the impact made by fiscal policy, i.e. crowding-out effect.*

- How does this relate to the shape of IS curve:
  - From the IS equation, $Y = \frac{1}{1-b(1-t)}[a - bT_0 + I_0 - I_1 r + G_0]$
  - Mechanically, an increase in government expenditure is typically associated with an increase in income.
  - Following the initial increase in income, this would cause an increase in " r" which generates the crowding out effect of government expenditure, dampening the initial effect of G.
    - $r = \frac{1}{L_2}\left(L_0 + L_1 Y - \frac{M_0^S}{P}\right)$
  - Fiscal policy produces the strongest impact when *it does not alter market interest rate*, i.e. $L_2 = \infty$.
  - Under the less extreme case, fiscal policy is highly effective if the crowding effect is weak, i.e. when we have the high value L_2.
    - This is consistent with the situation when LM curve is flat.

Crowding out effect in the graphical illustration



Chapter 4: Matrix algebra and its application

4.1 Notion of matrix and Basic operation

Matrix definition:

A matrix consisting of m horizontal rows and n vertical columns is called an ***m×n matrix*** or a ***matrix of size m×n***.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{12} & \dots & a_{2n} \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

- For the entry a_{ij}, we call i the row subscript and j the column subscript.

Row Matrix or Row Vector

- *The case for which there is* single row *in the matrix, i.e.* $m = 1$.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \end{bmatrix}$$

Column Matrix or Column Vector

- *The case for which there is* single column *in the matrix, i.e.* $n = 1$.

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ \\ a_{m1} \end{bmatrix}$$

Equality of matrices

Matrices A = [a_{ij}] and B = [b_{ij}] are equal if they have

(i) the same size and

(ii) $a_{ij} = b_{ij}$ for each i and j.

Transpose of a Matrix

A transpose matrix is denoted by A^T or A'

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$ ➔ A^T is

4.2 Some special matrices

Zero Matrix

- Can be called "*Null Matrix*"
- An m x n matrix with all entries or elements **are** zero.

Square Matrix

- A square matrix is a special type of matrix that has the same number of row as column, i.e. m = n.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

- Note that transpose of square matrix continues to be a square matrix.

Diagonal Matrix

- A special type of square matrix whose off-diagonal elements are entirely zero.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

- Note that the case for which such matrix has all zero-entries is called *"Zero Matrix"*

- The case for which all main diagonal elements are unit is called *"Identity Matrix"* (To be discussed later!)

Triangular Matrix

- Upper Triangular
  - *A square matrix that contains only zero for all entries below its main diagonal.*

- Lower Triangular
  - *A square matrix that contains only zero for all entries above its main diagonal.*

$$A = \begin{bmatrix} 5 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}$$

4.3 Basic operations of matrix

Addition/Subtraction

Sum A + B is the $m \times n$ matrix obtained by adding corresponding entries of **A** and **B**.

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ (commutative property)
2. $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$ (associative property)
3. $\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$ (identity property)

<u>**Example 4.B:**</u>

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 7 & -2 \\ -6 & 4 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1+7 & 4-2 \\ 2-6 & 5+4 \\ 3+3 & 6+0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Example 4.C: Individual and aggregation

Demand for the consumers is

$$D_1 = \begin{bmatrix} 3 & 2 & 5 \end{bmatrix} \qquad D_2 = \begin{bmatrix} 0 & 17 & 1 \end{bmatrix} \qquad D_3 = \begin{bmatrix} 4 & 6 & 12 \end{bmatrix}$$

For the industries is

$$D_C = \begin{bmatrix} 0 & 1 & 4 \end{bmatrix} \qquad D_E = \begin{bmatrix} 20 & 0 & 8 \end{bmatrix} \qquad D_S = \begin{bmatrix} 30 & 5 & 0 \end{bmatrix}$$

What is the total demand for consumers and the industries?

4.4 Multiplication

4.4.1) Scalar multiplication

1. $k(\mathbf{A}+\mathbf{B}) = k\mathbf{A} + k\mathbf{B}$
2. $(k+l)\mathbf{A} = k\mathbf{A} + l\mathbf{A}$
3. $k(l\mathbf{A}) = (kl)\mathbf{A}$
4. $0\mathbf{A} = \mathbf{O}$
5. $k\mathbf{O} = \mathbf{O}$

$$(\mathbf{A}+\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$$
$$(k\mathbf{A})^{\mathrm{T}} = k\mathbf{A}^{\mathrm{T}}$$

Example 4.D:

$2)\ (B-C)^T = B^T - C^T$

$$A=\begin{bmatrix}1 & 2\\0 & -1\\2 & 0\end{bmatrix},\qquad B=\begin{bmatrix}1 & 3\\4 & -1\end{bmatrix}$$

$$C=\begin{bmatrix}1 & 0\\1 & 2\end{bmatrix},\qquad D=\begin{bmatrix}1 & 2 & -1\\1 & 0 & 2\end{bmatrix}$$

Compute the followings

$6)\ (D-2A^T)^T = D^T-(2A^T)^T$

$= D^T - 2A$

1. $3A^T + D$
2. $(B-C)^T$
3. $2B^T - 3C^T$
4. $2B + B^T$
5. $C^T - D$
6. $(D-2A^T)^T$

Exercise 4.A:

Find the value of x and y that makes the equality condition of the two matrices hold.

$$x\begin{bmatrix}2\\0\\3\end{bmatrix}+2\begin{bmatrix}-1\\0\\6\end{bmatrix}+y\begin{bmatrix}0\\2\\-3\end{bmatrix}=\begin{bmatrix}8\\4\\3x+12-3y\end{bmatrix}$$

4.4.2) Matrix multiplication

- Let A be m x n matrix and B be an p x q matrix. Then the product AB exists if $n = p$. (#Column A = #row B!)
    - Given the necessary condition, the product results in an m x q matrix C.

E.g. 1 $\mathbf{A} = 3 \times 5$ matrix *and* $\mathbf{B} = 5 \times 3$ matrix ➔
$\mathbf{AB} = 3 \times 3$ matrix *but* $\mathbf{BA} = 5 \times 5$ matrix.

E.g. 2 $\mathbf{A} = 3 \times 5$ matrix *and* $\mathbf{B} = 7 \times 3$ matrix ➔
$\mathbf{AB} =$ undefined *but* $\mathbf{AB} = 7 \times 5$ matrix.

- The entry of matrix C in row i and column j is obtained as follows: sum the products formed by multiplying, in order, each entry (that is, first, second, etc.) in row i of A by the *"corresponding"* entry (that is first, second, etc.) in column j of B.

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{in} b_{nj}$$

- In fact, it's just pairwise inner products. (*dot products*)
    - Inner product What is the inner product?

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times 3} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}_{3\times 1} \Rightarrow [\quad]_{1\times 1}$$

Example 4.E

$$\begin{bmatrix} 2 & -4 & 2 \\ 0 & 1 & -3 \end{bmatrix}_{2\times 3} \begin{bmatrix} 2 & 1 \\ 0 & 4 \\ 2 & 2 \end{bmatrix}_{3\times 2} \Rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$$

Example 4.D

Given the price (of input) and the quantities(of input) vector,

$$\boldsymbol{P} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \textbf{and} \quad \boldsymbol{Q} = \begin{bmatrix} 7 \\ 5 \\ 11 \end{bmatrix}$$

What is the matrix representation for total cost?

The total cost is equal to

Example 4.E Given Ax = c as the form of the matrix representation of system of linear equation, write the below system in the matrix form.

$6x + 3y + z = 22$

$x + 4y - 2z = 12$

$4x - y + 5z = 10$

$3x + 2y = 7$

$5x - 7y + z = 0$

4.5 System of simultaneous equations and Matrix approach to the solution

A system of linear equations typically comprises of **M equations and N unknown.** The vector of N-unknown of "x" that satisfies all the M equations *simultaneously* is called the solution of the system.

Our task is to solve for the N-unknown variables!

If exists, matrix procedure allows us to solve for the solution under any arbitrary numbers of M and N. However, we would focus only the case when M =N. That is, we concentrate on the *just-identified* system of equation.

Just-(exactly) identified system

System of linear equations: exact identified system

N-Unknown variables

Equation 1: $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N = d_1$

Equation 2: $a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N = d_2$

$|$

Equation N: $a_{N1}x_1 + a_{N2}x_2 + \cdots + a_{NN}x_N = d_N$

Solution to the N-unknown variables can be derived by two methods.

- An inverse matrix approach
- The Cramer's rule

4.5.1) Inverse matrix method

To begin the analysis, let's rewrite the above system of equations in terms of the following matrix representation

$$Ax = d$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_N \end{bmatrix}$$

A = matrix of coefficients that are associated with unknown variables

d = matrix of coefficients that are independent of unknown variables

x = vector of unknown variables
= endogenous variables

To obtain the solution, pre-multiply A with an inverse of A.

$$A^{-1}Ax = A^{-1}d$$

By the definition of inverse matrix A, it must be the case that

$$A^{-1}A = I_N$$

As a result, we yield that

$$x = A^{-1}d$$

Definition: Finding the inverse of A

$$A^{-1} = \frac{adj(A)}{\det(A)}$$

where adjoint of A, i.e. adj(A), is an n x n matrix transpose of the matrix whose entries are each *corresponding cofactor*. i.e. **cofactor matrix of A.**

To understand everything, let's refresh our knowledge of determinant, minor, and then cofactor. First things first.

Determinant

For N = 2, determinant of any square matrix A can be calculated by:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow \det(A) = |A| = a_{11}a_{22} - a_{21}a_{12}$$

Example 4.H

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 6 \\ 8 & 24 \end{bmatrix}$$

Definition: A is said to a *singular* matrix if det(A) = 0

Theorem:
If A is a *singular* matrix, the just-identified system contains infinite number of solution.

<u>Laplace expansion</u>

To find the determinant of any square matrix A of order N (N>2), select any row (or column) of A and *multiply each entry in the row (column) by its <u>cofactor</u>*. The *sum of these products* is defined to be the determinant of A and is called a *determinant of order n.*

What is the Cofactor?

The corresponding value of cofactor to the i-th column and j-th column matrix is

$$C_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is its associated minor.

Definition: Minor

M_{ij} is the *subdeterminant* from deleting the i-th row and j-th column.
Determinant of sub-matrix

Example 4.F

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$M_{13} =$ Deleting row# 1 and Column# 3

$M_{22} =$ Deleting row# 2 and Column# 2

Definition: Cofactor matrix of A or $cof(A) = [c_{ij}]$ is the collection of the cofactor value

Example 4.G: Find the invest matrix of

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 0 & -5 \\ 2 & 1 & 1 \end{bmatrix}$$

Step 1: Calculate all Minors, and cofactor matrix

M11	M12	M13
M21	M22	M23
M31	M32	M33

Step 2: Calculate det(A)

Step 3 : Adj(A) / det (A)

4.5.2) Cramer's rule

The linear system $\boldsymbol{Ax} = \boldsymbol{d}$ has a solution given by the *Cramer's formulas:*

$$x_i = \frac{\det(A_i)}{\det(A)}, \quad i = 1,2,3,\dots,N$$

where (i) $\boldsymbol{x_i}$ are the unknowns of the system or the i-th entries of x, and

(ii) the matrix A_i is obtained from $\boldsymbol{A}$ by replacing the ***i-th*** column by the vector ***d***

Example 4.J: Find the solution using the Cramer's rule

$$\begin{pmatrix} 1 & 2 & 0 \\ -1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

4.6 Application examples

Note: Mapping the linear economic model in a matrix form

Example 4.K: Using the form Ax = d, write the following market model in the matrix form.

$$Q_d = Q_s$$

$$Q_d = a - bP + cY$$

$$Q_s = e + fP + gW$$

when a, b, c, e, f, g are (non-negative) parameters. Y = income and W = weather condition.

Example 4.L

Consider a simple macroeconomic model.

$C = a + bY_d; \quad 0 < b < 1$

$I = I_a + iY; \quad 0 < i < 1$

$G = G_0$

$T = T_0 + tY; \quad 0 < t < 1$

$R = R_0$

$Y_d = Y - T + R$

where R is the government transfer and G is the government purchase. All the remainings are defined as usual.

- State all the endogenous variables in the model

- Is the above system solvable?

- Simplify the system to a 3-variable system that includes Y, C and I.

- Rewrite the system of equations in the matrix form and solve for all the three variables using the Cramer's rule.

Chapter 5: Single-variable (Univariate) optimization: theory and applications

Optimization theory is a mathematical tool developed to solve for the solution of the following maximization problem,

$$\max_{x} f(x) \quad , \quad x \in D,$$

or the minimization problem

$$\min_{x} f(x) \quad , \quad x \in D,$$

where "x" is the argument that we choose.

- Optimization theory has been extensively applied into economics analysis.
  - We've seen before that equilibrium economic models include several behavioral equations that capture behavior of groups/persons engaged in the model.
  - All these behavioral equations are actually *derived* from *the first principle* that agents are rational and choose for *optimal decision*, resulting in the behavior that can be captured by the proposed function.
    - Market model
      - Supply equation is the result of profit-maximizing decision of firm.
      - Demand equation is the result of maximizing utility, subject to budget set. (You've seen this, but not in a mathematical version. Recall "IC: indifferent curve" and "BS: budget set")
    - National income model
      - Aggregate consumption function is derived from the aggregation of individual consumption function which is the result from intertemporal consumption problem.

5.1 Basic calculus: derivative

5.1.1) What is derivative?

- Derivative measures the responsiveness of "y" attributed to "x".

For example, consider a function with $y = f(x)$.

- Derivative of y with respect to x is denoted by $\frac{dy}{dx}$ *or* $f'(x)$.
- Very much like to the concept of *delta change*, derivative considers the case for a small change.

By the definition, mathematical notion for derivative is *limiting case* for

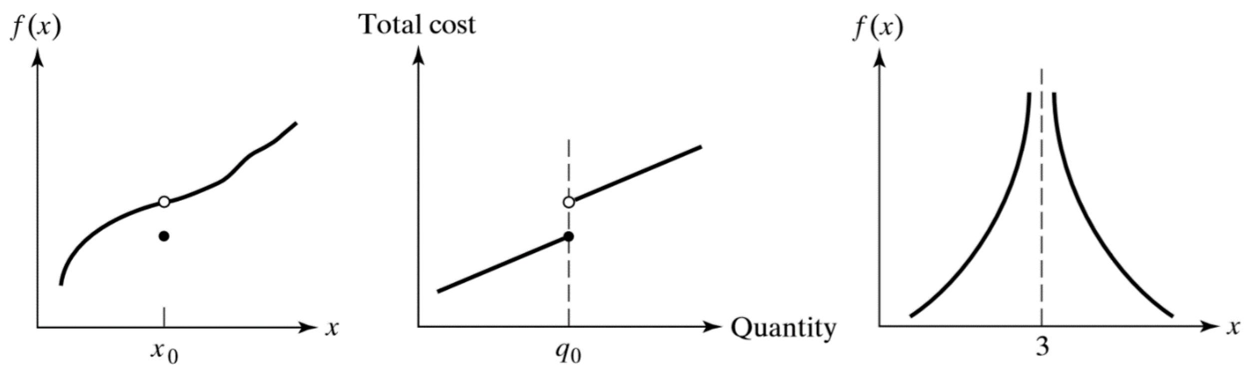
$$\frac{dy}{dx} = \frac{\Delta y}{\Delta x}; (\Delta x \approx 0)$$

Formally stated, the notion of derivative is given by,

$$\begin{aligned}\frac{dy}{dx} = f'(x_0) &= \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h} \\ &= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} \\ &= \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\end{aligned}$$

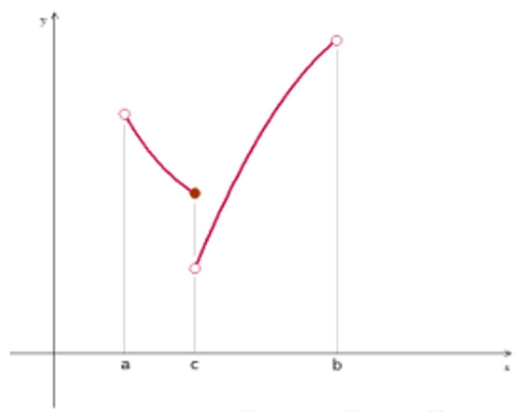
We note that *a function is differentiable at* $x = x_0$ when $\frac{dy}{dx}$ at $x = x_0$ exists. In general, a function is not always warranted to be differentiable. Two possible cases are the followings.

- Function is Not *continuous*

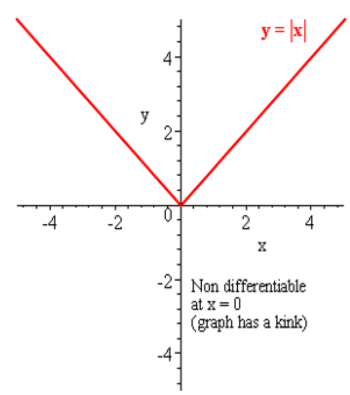


- Function is Not *smooth: graph has a kink!*

Not continuity



Not smooth ➔ non-differentiable.

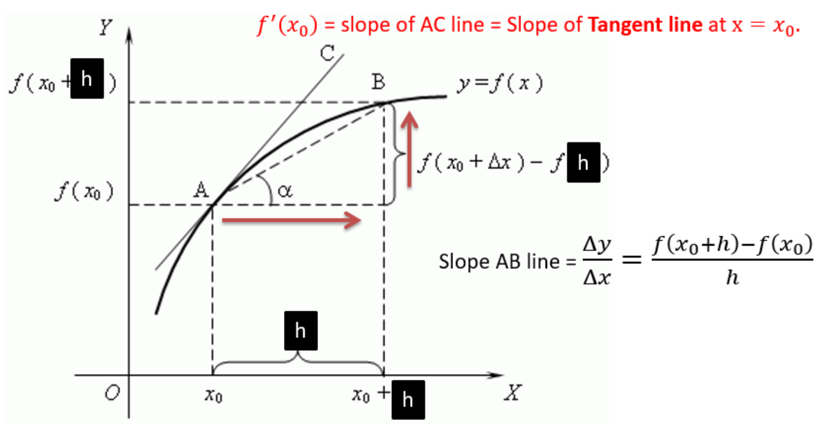


Mathematically, we call a function that can be differentiable for the *entire domain* of "f" as a *differentiable function*.

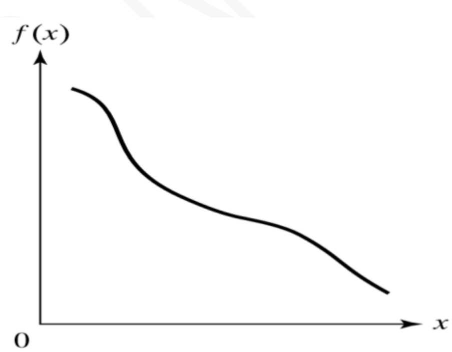
5.1.2) What does the derivative tell us about?

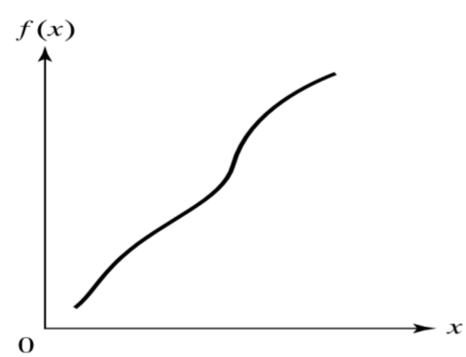
- Since derivative measures the responsiveness when change is very small, its basic idea is related to the concept of "*instantaneous* rate of change" and "*slope of the tangent line*".

Derivative and Slope in graphical illustration.

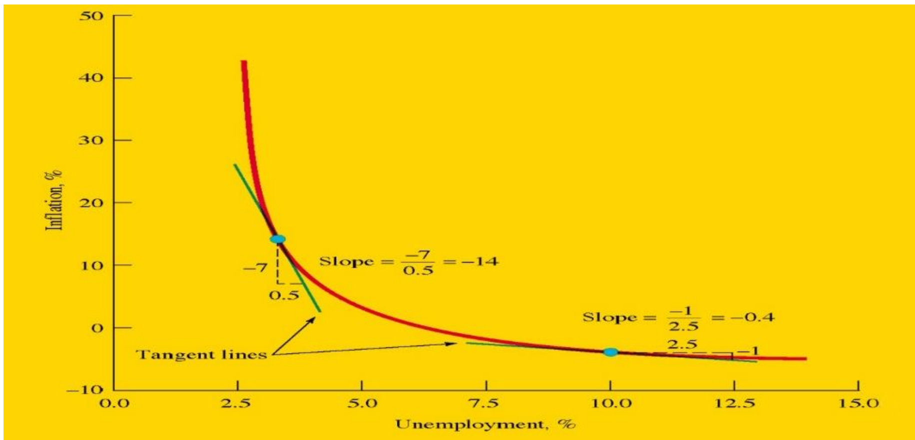


- Interpretably, derivative tells us both "*direction*" and "*magnitude*" of change in y that is attributed to change in certain direction of x.
    - Direction: One can denote the characteristic of function using the sign of derivative; *Increasing vs. decreasing function*





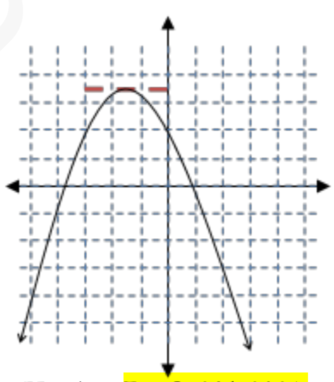
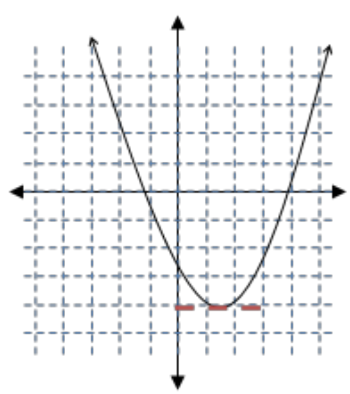
- *Magnitude: How big is the change?*
  - The below figure gives an example of a downward sloping curve; the figure graphs the relationship between inflation and unemployment rate. (Do you know the name of this curve?)



  - Notice two things. (i) Slope of the tangent line is negative, and (ii) Slope of tangent line changes with respect to the unemployment rate; *inflation become more sensitive to unemployment rate at the low unemployment rate than at the high unemployment rate.*

- Derivative is too used as an input for the *approximation* problem.
  - To be more precise, this is known as the *differential approximation* problem.

- Derivative is also helpful for giving some information about the nature of the extreme point.
  - First-order necessary condition suggests that local optimizer is mostly the stationary point of the function, i.e. *the point whose tangent line is paralleled to the horizontal axis.*

5.2) Rule of differentiation (All discussed in MA216)

Differentiating a function can be easily done using the formula; one needs not to start from the limit concept.

RULE 1 Derivative of a Constant: $\frac{dc}{dx} = 0$

RULE 2 Derivative of x^n: $\frac{dx^n}{dx} = nx^{n-1}$

RULE 3 Exponential $\frac{d(b^x)}{dx} = b^x \ln b$

RULE 4 Logarithm $\frac{d(\log_b x)}{dx} = \frac{1}{x} * \frac{1}{\ln b}$

RULE 5 Sum or Difference Rule $\frac{\mathrm{d}[\mathrm{f}(\mathrm{x}) \pm \mathrm{g}(\mathrm{x})]}{\mathrm{dx}} = \mathrm{f}'(\mathrm{x}) \pm \mathrm{g}'(\mathrm{x})$

RULE 6 Constant Factor Rule: $\frac{d[cf(x)]}{dx} = cf'(x)$

RULE 7 Product rule $\frac{d(f*g)}{dx} = f'(x)g(x) + g'(x)f(x)$

$$
\begin{aligned}
. \ y &= (x^2+1)(x^4+4x+1) \\
\frac{dy}{dx} &= (x^2+1)\tfrac{d}{dx}(x^4+4x+1) + (x^4+4x+1)\tfrac{d}{dx}(x^2+1) \\
&= (x^2+1)(4x^3+4)(x^4+4x+1)(2x)
\end{aligned}
$$

RULE 8 Quotient rule $\frac{d\left(\frac{f}{g}\right)}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

$$
\begin{aligned}
\text{ex. } y &= \frac{x^2+2x+3}{x^3+7} \\
\frac{dy}{dx} &= \frac{(x^3+7)\frac{d}{dx}(x^2+2x+3) - (x^2+2x+3)\frac{d}{dx}(x^3+7)}{(x^3+7)^2} \\
&= \frac{(x^3+7)(2x+3) - (x^2+2x+3)(3x^2)}{(x^3+7)^2}
\end{aligned}
$$

RULE 9 Chain rule $\frac{df(g(x))}{dx} = f'(g(x)) * g'(x)$

$$\begin{aligned} \text{ex. } y &= ln(x^2+1) \\ u &= x^2+1 \\ \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot \frac{du}{dx} = \frac{1}{x^2+1} \cdot 2x \end{aligned}$$

Each function might be differentiated for *a number of repeated rounds.* This brings in the concept of higher-order derivative.

- 2nd order ➔ $f''(x) = \frac{df'(x)}{dx} = \frac{d^2f(x)}{dx^2}$
- 3rd order ➔ $f^3(x) = \frac{df''(x)}{dx} = \frac{d^3f(x)}{dx^3}$
- nth order ➔ $f^n(x) = \frac{df^{n-1}(x)}{dx} = \frac{d^nf(x)}{dx^n}$

Example: find $\frac{d^2y}{dx^2}$ when $y = \ln(x^2+1)$

$$\frac{dy}{dx} = \frac{1}{x^2+1} \cdot \frac{d(x^2+1)}{dx} = \frac{1}{x^2+1} \cdot (2x) = \frac{2x}{x^2+1}$$

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{2x}{x^2+1}\right)}{dx} = \frac{(x^2+1)\left(\frac{d(2x)}{dx}\right) - 2x \cdot \frac{d(x^2+1)}{dx}}{(x^2+1)^2}$$

In general, a function that can be *successively* differentiated for *"n times"* is commonly called **"C^N function"**.

Example 5.B Make sure you are familiar with all these!

a. $F(x) = 3x^5 + \sqrt{x}$

b. $f(z) = \frac{z^4}{4} - \frac{5}{z^{1/3}}$

c. $y = 6x^3 - 2x^2 + 7x - 8$

$$F(x)=(x^2+3x)(4x+5)$$

$$F(x)=\frac{4x^2+3}{2x-1}$$

$$y=(x^2-4)^5(3x+5)^4$$

<u>Derivative for some economics functions</u>

C(y) = consumption function

- Marginal propensity to consume = $MPC = \frac{dc}{dy}$

C(q) = cost function

- Average cost function: $AC(q) = \frac{C(q)}{q}$
- Marginal cost function: $MC(q) = \frac{dc}{dq} = C'(q)$

R(q) = revenue function

- By the construction, R(q) = p*q
- Therefore, average revenue is then $AR(q) = \frac{R(q)}{q}$. This is equal to "p".
- Marginal revenue is $\frac{dR}{dq} = R'(q)$.

Elasticity:

$$\frac{\Delta y}{\Delta x} * \frac{x}{y} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} * \frac{x}{y} = \frac{\frac{dy}{y}}{\frac{dx}{x}} = \frac{d(\ln y)}{d(\ln x)}$$

Example 5.D:

Suppose that consumption function takes the form: $C = 10 + 0.1Y + 6\sqrt{Y}$

Find the value of MPC when Y = 100 and Y = 1000.

Example 5.D:

If a manufacturer's average-cost equation is

$$\bar{c} = 0.1q^2 - 0.02q + 5 + \frac{500}{q}$$

Find the marginal-cost function. What is the marginal cost when 50 units are produced?

Example 5.E

Given the demand function, $p = \frac{k}{q}$ where $k > 0$ and $q > 0$

Show that the demand has a constant unit elasticity

5.3) Differential and approximation

Differential

Derivative/differentiation:

$$\frac{dy}{dx} = f'(x_0)$$

Differential in y is given by:

$$dy = f'(x_0)dx$$

28

Differential is the linear approximation around x_0. Why? And How?

Example 5.F: Approximate the value of y for each of these values of x.

$$y = \ln(1 + x)$$

$x = 0$ ➔

$x = 1$ ➔

$x = 0.01$ ➔

Accuracy of the approximation

- Accuracy of the approximation increases as Δx approaches zero.
- When Δx gets bigger in the size, accuracy of the linear approximation gets worse.
- To fix the problem, one needs to rely on the **higher order** approximation: *the Taylor's approximation.*

Differential: *Taylor's approximation*

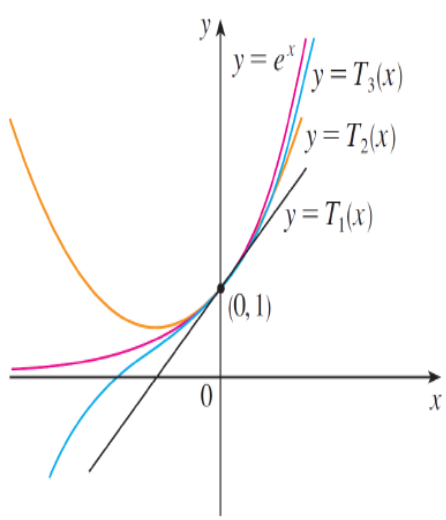
- $$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \frac{1}{3!} f'''(x_0)(x - x_0)^3 + \frac{1}{4!} f^4(x_0)(x - x_0)^4 + \cdots + \frac{1}{n!} f^n(x_0)(x - x_0)^n$$

! = Factorial

N-th ordered polynomial term

N-th ordered derivative

Example: $y = e^x$, approximate around x = 0



	$x = 0.2$	$x = 3.0$
$T_2(x)$	1.220000	8.500000
$T_4(x)$	1.221400	16.375000
$T_6(x)$	1.221403	19.412500
$T_8(x)$	1.221403	20.009152
$T_{10}(x)$	1.221403	20.079665
e^x	1.221403	20.085537

5.3 Extreme-point problem

5.3.1) Statement of the problem

Suppose $f(x)$ is an objective/target/return/criteria function.

Mathematical formulation of the extreme point problem is given by the maximization problem,

$$\max_{x} f(x) \quad , \ x \in D,$$

or the minimization problem

$$\min_{x} f(x) \quad , \ x \in D,$$

5.3.2) Types of extreme point

- *Local* extreme point (optimizer)
    - For all the "x" locating in the neighborhood of x*, the x* is a local optimizer if

      *Neighborhood of x** = $B(x^*) = \{x \mid |x - x^*| < \varepsilon \}$ }

        - Local (relative) maximizer: $f(x^*) \geq f(x)$.
        - Local (relative) minimizer: $f(x^*) \leq f(x)$.

- *Global* extreme point (optimizer)
    - For all the "X" in the domain defining "f" ($D = D_f$), the x* is the global optimizer if
        - Global (abosolute) maximizer: $f(x^*) \geq f(x)$
        - Global (abosolute) mininmizer: $f(x^*) \leq f(x)$

<u>Some technical notes:</u>

- In general, optimizer may *not* exist. (When can the existence be warranted: continuous function with *<u>compact</u>* domain sets.)

- Suppose it does exist, *how to locate the points*?
  - Differential calculus provides some important toolkits for searching the local optimizer.
  - In economics, we try to locate the *global optimizer*!! So, the problem itself is very hard because the toolkit is only applicable for detecting local optimizer.

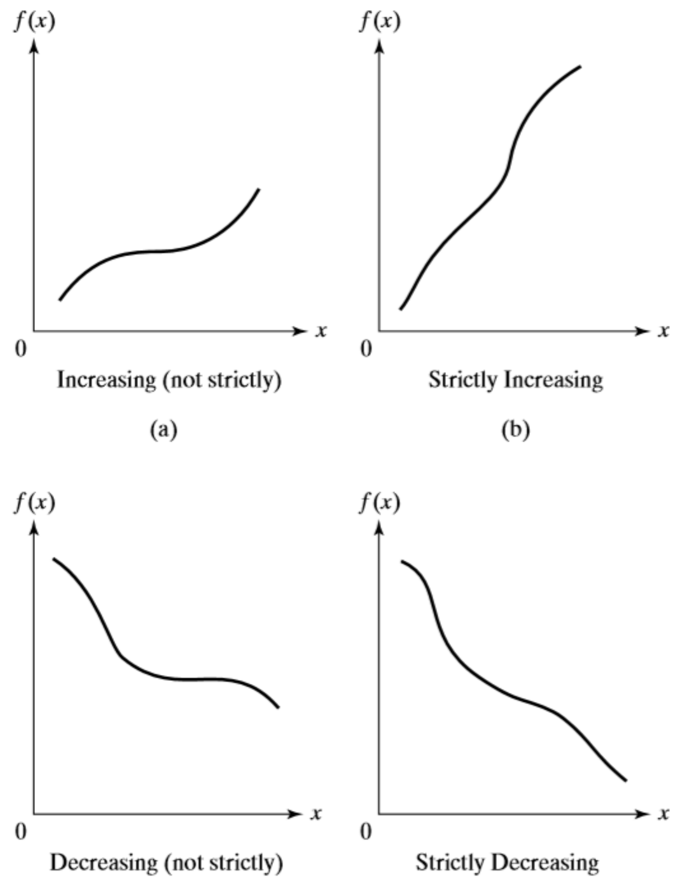
- However, under some *regularity conditions* on the given function and the domain set in consideration, local optimizer can be warranted to be the global optimizer. (What are the conditions?)

- To understand the tool that we need for solving the optimization problem, we need to develop some basic understanding on the characteristics of function implied by the derivative.

5.3.3) Characteristic of functions by derivative

5.3.3a) Increasing vs decreasing function

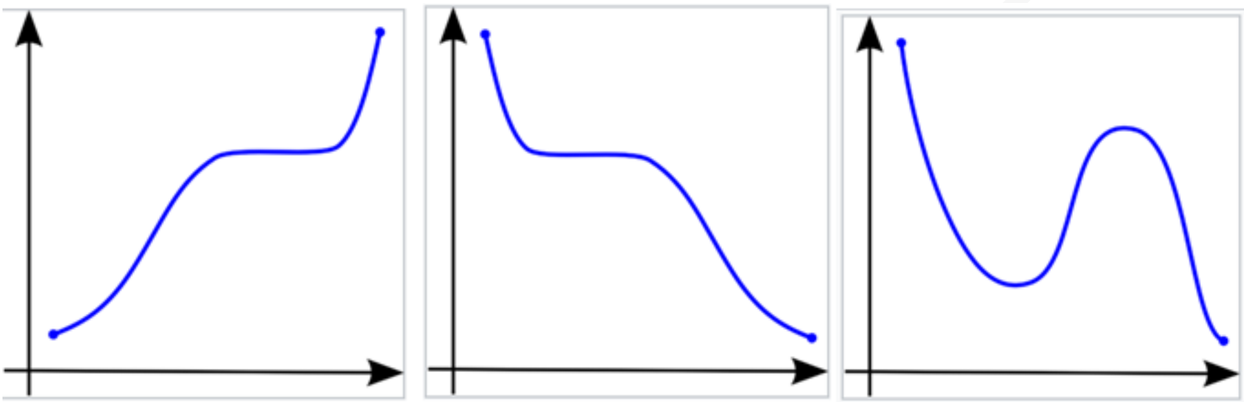
- **(Weakly) Increasing**: A function is said to be increasing if and only if for any $x_1 > x_2$, we have that $f(x_1) \geq f(x_2)$, and vice versa.
- **(Weakly) Decreasing**: A function is said to be decreasing if and only if for any $x_1 > x_2$, we have that $f(x_1) \leq f(x_2)$, and vice versa.
- ***Strictly* increasing**: A function is said to be strictly increasing if and only if for any $x_1 > x_2$, we have that $f(x_1) > f(x_2)$, and vice versa.
- ***Strictly* decreasing**: A function is said to be strictly decreasing if and only if for any $x_1 > x_2$, we have that $f(x_1) < f(x_2)$, and vice versa.



- By the application of derivative, one can show that the increasing/decreasing property is equivalent to having $f'(x) \geq 0$ or $f'(x) \leq 0$
  - ***Strictly* increasing** : $f'(x) > 0$
  - ***Strictly* decreasing** : $f'(x) < 0$

Note that a function can be increasing in some parts of the domain, while being a decreasing one in other parts. That is, *the sign of derivatives is mixed.* Mathetically, when a function is always non-negative (or non-positive) for the entire domain of "f", we call the function as a ***monotonic*** increasing (decreasing) function.

Example



(a) monotone increasing (b) monotone decreasing (c) non-monotonic function

5.3.3b) Concave and convex function

Convexity and concavity generally inform us about the *curvature* of a function. To understand this, we start out with some "formal" definitions, and then follow by the more customarily used conditions implied by the *sign of second derivative.*

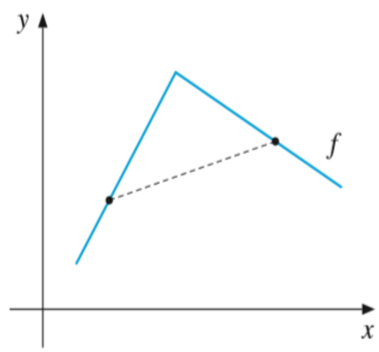
Defintion: Convex v.s. Concave

- **Concave function:** For an interval $[x_1, x_2]$, a function is said to be concave if we have that $f(tx_1 + (1-t)x_2) \geq tf(x_1) + (1-t)f(x_2)$ for any $t \in (0,1)$
    - *Strictly* concave: the condition holds with strict inequality.

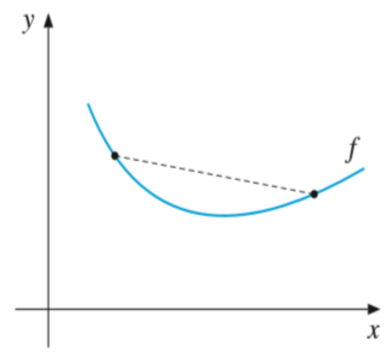
- **Convex function:** For an interval $[x_1, x_2]$, a function is said to be convex if we have that $f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$ for any $t \in (0,1)$
    - *Strictly* convex: the condition holds with strict inequality.

To ease the understanding, we first note that $tf(x_1) + (1-t)f(x_2)$ can be visually represented by a straight line, which is called the *secant (segment) line.* The secant line could be visually constructed by drawing a straight line that connects between x_1 and x_2. Graphically, a function "f" is called concave (convex) if the secant line for any two points on the graph is below (above) the graph, or on the graph.

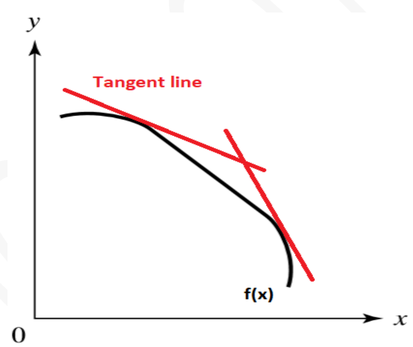
- When x_1 and x_2 are the same point, the secant line becomes a tangent line. Thus, it evideintly follwos that for a concave (convex) function, the tagent line lies above (underneath) the graph.



f: concave

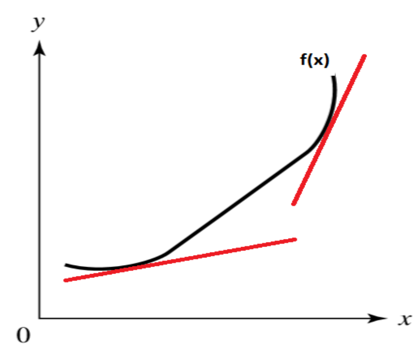


f: (strictly) convex



Concave

(a)

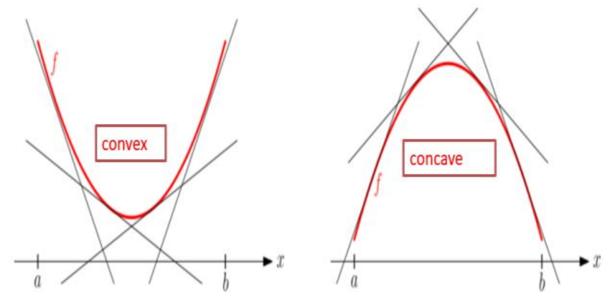


Convex

(b)

Too see if the function is convex or concave, one would rarely check for its property from the formal statement above. In practice, it is more convenient to check the sign of derivative. Mathemtically, the following statements are equivalent,

- **Concavity**:
  - Slope of f(x) is *weakly* decreases as X increases.
    - $f'(x)$ is a decreasing function
  - The second-order derivative is non-positive, i.e. $f''(x) \leq 0$.
- **Convexity**:
  - Slope of f(x) is *weakly* decreases as X increases.
    - $f'(x)$ is an increasing function
  - The second-order derivative is non-positive, i.e. $f''(x) \geq 0$.



all straight lines are tangents to f and lies below f where f' is increasing

all straight lines are tangents to f and lies above f and f' is decreasing

- **Note:** For strictly concave (convex), the conditions hold with "*strict*" ineqaulity.

5.3.4) Local optimizers

- Differential Calculus only provides us a tool set for solving *local extreme point*.

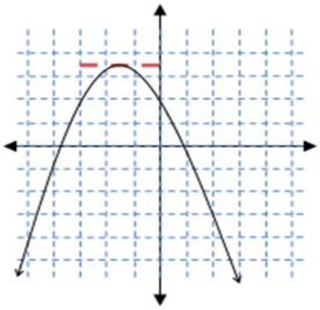
  - If one wants to solve for global extreme points, one needs to check all possible local extreme points, and boundary values of the domain set that defines the function.

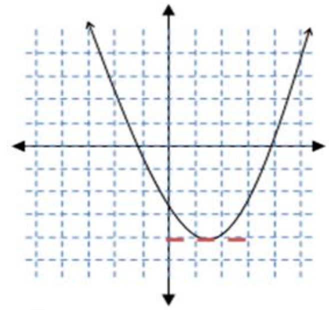
  - Under some regularities conditions, *local extreme point is warranted to be global extreme point*.

- Mathematician notices some properties of the local optimizers for any twice-differentiable function Then, they construct an important theorem that characterizes the property of location optimizers. This is known under *first-order* and *second-order necessary condition*.

Theorem: *First order necessary condition*

Suppose that "f" is a *differentiable function*. If x^* is a local optimizer, then $f'(x^*) = 0$. That is, x^* is also a stationary point.





Theorem: *Second-order necessary condition*

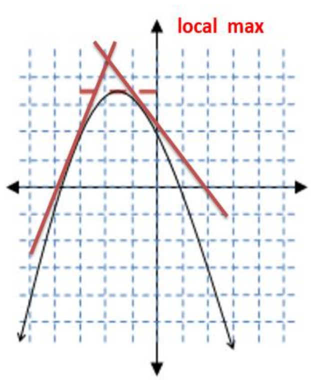
Suppose that *"f"* is twice-differentiable function. If x^* is local maximizer (minimizer), then $f''(x^*) < (>)\ 0$. That is, x^* is located under the concave (convex) interval of the function.

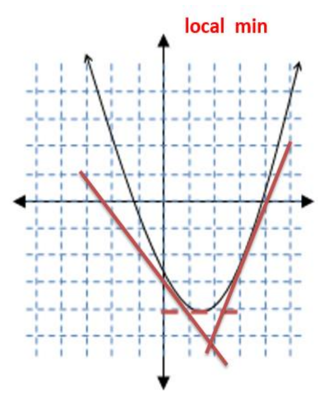
Proof:

By the Taylor's approximation, we know that

$$f(x) = f(x^*) + f'(x^*)(x - x^*) + \frac{1}{2}f''(x^*)(x - x^*)^2.$$

Note that the approximation should be very accurate because we are considering the value of x in the neighborhood of x*





From the first-order necessary condition, we know that local optimizer is a stationary point, and hence $f'(x^*) = 0$. Therefore, the approximate form can be further simplified into:

$$f(x) - f(x^*) = \frac{1}{2}f''(x^*)(x - x^*)^2$$

Suppose that x^* is a local minimizer, the above equation would imply that

$$\frac{1}{2}f''(x^*)(x - x^*)^2 > 0$$

Since $(x - x^*)^2$ is always positive, the term $f''(x^*)$ must be positive; this is to ensure that above inequality holds. Consequently, the local minimizer, x^*, must be located under the convex region of the function

Sufficient theorem:

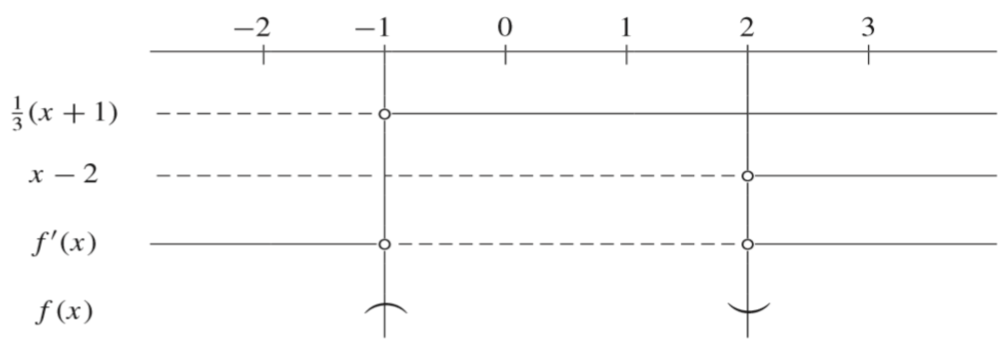
From the conditions discussed above, one can normally locate the local optimizer using *two related methods (cooking recipes)*, ecach of which contains two-step verification.

- **Method 1:** *First-derivative test*
  - Step 1: Check for the stationary points
  - Step 2: Distinguish the type of optimizer by checking the sign of first derivative, and they way it changes.
    - Maximum: positive sign turns into negative sign
    - Minimum: negative sign turns into positive sign

Example

Classify the stationary points of $f(x) = \frac{1}{9}x^3 - \frac{1}{6}x^2 - \frac{2}{3}x + 1$.

Solution: We get $f'(x) = \frac{1}{3}(x+1)(x-2)$, so $x = -1$ and $x = 2$ are the stationary points. The sign diagram for $f'(x)$ is:



We conclude from this sign diagram that $x = -1$ is a local maximum point whereas $x = 2$ is a local minimum point.

- **Method 2:** *Second-derivative test*
    - Step 1: check for the stationary points
    - Step 2: Check for the sign of second-derivative (convexity/concavity)
        - Maximizer: $f''(x^*) < 0$ (decreasing slope)
        - Minimizer: $f''(x^*) > 0$ (increasing slope)

Example (cont.)

From the above example, $f''(x) = \frac{2}{3}x - \frac{1}{3}$.

This yields $f''(-1) = -1 < 0$ and $f''(2) = 1 > 0$

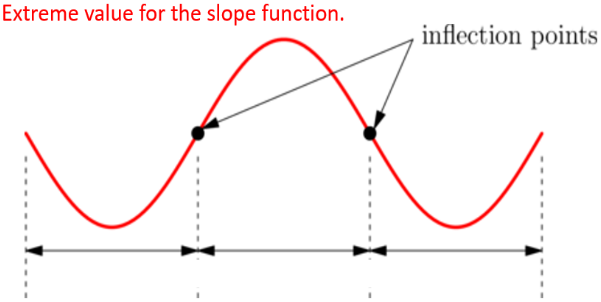
Example 5.G Locate all the local optimizers of $y = x^3 - 3x$. Also, state the type of local optimizer that you calculate.

5.3.5) Inflection point

Definition: $x^{inflection}$ is said to be an inflection point when the curvature of the function changes from convex (concave) to concave (convex) at $x = x^{inflection}$.

- $f''\left(x^{inflection}\right) = 0$.
- Extreme value for the slope function

- Changing from concave to convex, and vice versa….
- At that point, $f''\left(x^{inflection}\right) = 0$.
- Extreme value for the slope function.



Example 5.H: Given $y = x^4 - 2x^2 + 1$, find local optimizers and inflection points

5.3.6) N-th derivative test

- In some cases, we might encounter into a situation that both first and second derviatve at x^* are equal to zero, i.e.$f'(x^*) = 0$ and $f''(x^*) = 0$

- Consider for example $y = x^5 + 1$
  - $f'(0) = 0$
    - X = 0 is a critical (stationary) value. So, one could think that this is a candidate for local optimizer.
  - $f''(0) = 0$
    - But, this is also a possible candidate for an inflection.

- **Then, what is it actually?**
- When the second-derivative test method fails to distinguish the type of stationary point in question, one can instead apply the so called Nth –derivative test, i.e. a generalized version of the second-derivative test.

Theorem:
For the x^* with $f'(x^*) = 0$, and other higher-order derivatives are all also zero, but **the N-th,**
- if N is an even number,
  - x^* is local max if $f^N(x^*) \neq 0 < 0.$
  - x^* is local min if $f^N(x^*) \neq 0 > 0.$
- If N is odd number, x^* is an inflection point.

Example 5.I: $y = x^5 + 1$ v.s. $y = x^4 + 1$

5.3.7) Global optimizers

- Normally, it's very hard to locate where the point is.
- Most numerical work usually ends up wrongfully yielding the local optimizer as the answer; this is wrong.
- Engineering heavily relies on this technique, and extensively develop tools for identifying/detecting the global optimizers under more robust conditions.

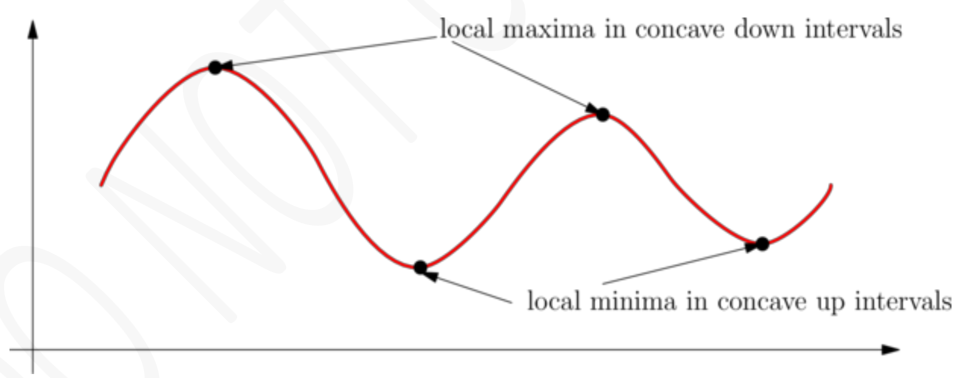
<u>Weistrass theorem</u>

Under a closed and bounded domain set of a continuous function, global optimizer is guaranteed to exist!

- It may *or* may not exist for the opened domain!

To find the global optimizers:

- Calculate the value of ***all the local optimizers*** and ***the two-value at the boundary***.



Example 5.J: $y = x^3 - 3x$; $x \in [-2, 2]$

Sufficient condition for the local optimizer as a global one

If we assume some nice properties of the function, a local optimizer automatically becomes a global optimizer. The theorem states as follows.

- If the function is *globally* concave, local maximizer is then global maximizer.
- If the function is *globally* convex, local minimizer is then global minimizer.

Definition: Local v.s. Global concavity

- Local concavity (convexity): A function is concave under certain interval of "x" that defines the function.
- Global concavity (convexity): A function is concave for every interval of "x" that defines the function.
  - *"Always concave or Always convex; everywhere"*

Example 5.K:

- $\mathrm{f}(x) = 2x^2 - 2x + 1$
  $f'(x) = 4x - 2 \Rightarrow \mathrm{f}''(x) = 4 > 0 \ ; \forall x$
  - globally convex
- From the theorem, the local optimizer, $x^* = \frac{1}{2}$ is warranted to be theglobal min.

Example 5.L:

- $\mathrm{f}(x) = -e^x + x$
  $\mathrm{f}'(\mathrm{x}) = -e^x + 1 \Rightarrow f''(x) = -e^x < 0 \ ; \ \forall x$
  - globally concave
- From the theorem, the local optimizer, $x^* = 0$ is warranted to be the global max.

5.4 Some economics applications

- Single- variable optimization is mostly applied to *firms' problem* and *market equibliurm analysis*.

- Recall some basic concepts in EE211 about *firms* and *market theory*.
  - We set out the analysis with firm's decision problem.
  - Then, we looked into the way in which firms' decision interacts with buyers (demand), and determine the equiblibrum in the market.

- This course will revist all those issuess using mathematics.

- The topics for which we discuss in class will be organized by market structures. The list is as follows,
  - Competitive market
    - Firm's optimization problem in competitive market
    - Individual supply curve.
    - Short-run competitive *equilibrium*
    - Long-run competitive equilbirum
  - Monopoly market
    - Firm's optimization problem in monopoly market

5.4.1) Competitive market and its equilibrium

- A model that predicts the equilibrium of market when *each individual takes price as given.*
  - Welfare theorist proves that this market results in the *first-best allocation*.
  - Biggest area of the "Welfare triangle"

- In previous part, we've seen before the technique to solve for a market equilibrium.
  - This requires the information about market demand curve and market supply curve.

- From EE211, we knew that both market demand and market supply are simply by-product of the aggregation of each individual demand and individual supply.

- Individual supply can be derived from the *profit maximization problem* (PMP)!

Profit maximizing problem (PMP):

Firm's problem is to choose for Q that maximizes profit.

$$\pi = R(Q) - C(Q)$$

$R(Q)$ differs across market structure.

- Can firm set its price? Is the firm taking price as given?
- Competitive firms: $R(Q) = PQ$
- Monopoly: $R(Q) = P(Q) * Q$

We will now consider the case for competitive firms. Our goal is to answer the following question: *how to pin down the optimal level of output that maximizes the profit function.*

Calculus of profit-maximizing output:

By applying the differential calculus, a candidate for profit-maximizing level of output satisfies the following *first-order condition*,

$$\pi'(Q^*) = 0 \Rightarrow R'(Q) - C'(Q) = 0$$

$$\Rightarrow MR(Q^*) = MC(Q^*) \quad ;$$

In economics, the above condition is referred to the *"optimality condition"*. That is, firms set Q where MR = MC.

Under a *"perfectly"* competitive market, *marginal revenue is equal to price*; firms cannot set price. The optimality condition is then simplified into,

$$\Rightarrow P = MC(Q^*)$$

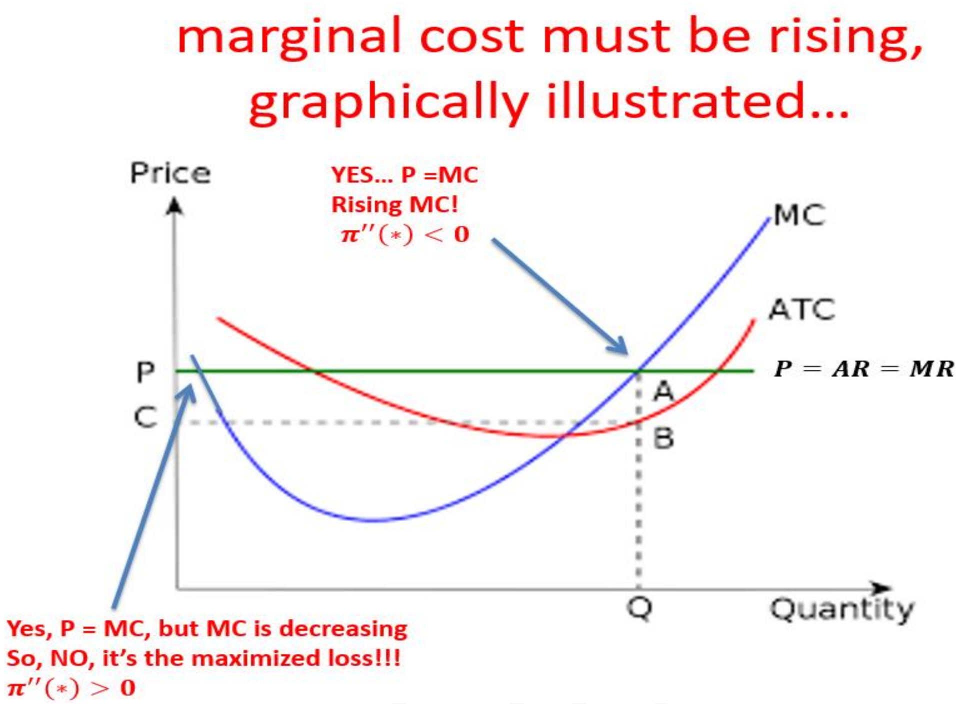
Mathematically, any Q that ensures the equality between MR and MC can be consider as a candidate of profit-maximizing output. But, which one is it? This requires the *second-derivative test*. The *second-order condition* requires that the second derivative of the profit function must be less than zero at Q where MR = MC.

$$\pi''(Q^*) < 0 \quad \Rightarrow MR'(Q^*) - MC'(Q^*) < 0$$

$$\Rightarrow -MC'(Q^*) < 0$$

$$\Rightarrow MC'(Q^*) > 0 \ ;$$

Graphically, what does the condition mean? Grpahically, we choose for *Q where MR=MC and marginal cost is rising*



Example 5.M: Suppose that unit-cost function takes the following equation:

$$AC(Q) = 0.2Q + 4 + \frac{400}{Q}$$

What is the level of profit-maximizing output when P = 8? What is the level of maximized profit?

The short-run supply curve

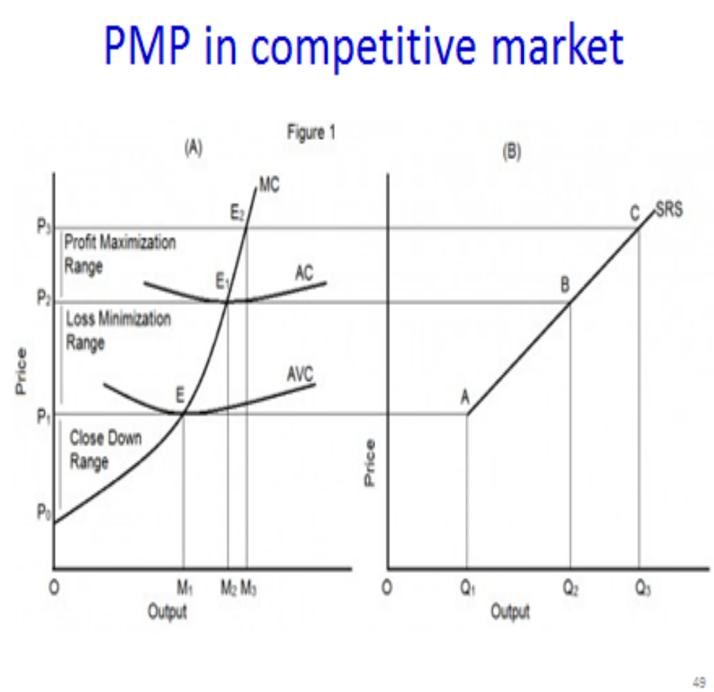
Definition: The *short-run* supply curve is the quantity-price schedule that results from the short-run profit maximization problem.

- Conceptually, we yield different level of optimal outputs for different prices.
    - From the above example, one would have obtained another Q if the price is different from 8.
- Mathematically, "optimal level" of output should be a function of market price! The derived relationship represents the notion of *firm's supply*.
    - Mathematically, we derived "Q" in terms of "P".

- Note that since it requires that we choose to the level of output where price is equated to marginal cost under the rising part of marginal cost, optimal quantityprice schedule (the supply relationship) must be increasing.

Active and inactive region: Shut-down

- Supply curve is NOT the entire curve of the marginal cost curve.
    - If price is dropping too low, firm might choose to temporarily stop its production.
    - This is called the *shut-down price.*
    - Shut-down price occurs when P = min(AVC) at that Q.
- Not to be confused with the break-even point, i.e. P = min(AC)



Example 5.M (contd): Use the information above and derive the supply equation. Properly characterize the region under which firm is active in the market.

Market equilibrium:

Definition: Given N firms in the market and market demand, short-run competitive equilibrium comprises of allocation "Q^e", $\{q_1^e, q_2^e, \ldots, q_N^e\}$ and "P^e" such that

(i) q_i^e is the profit-maximizing level of output at P^e.

(ii) $Q^e = q_1^e + q_2^e + \cdots + q_N^e$

(iii) P^e clears the market, i.e. $Q^e = demand$ at the equilibrium price

Equilibrium notion is somehow confusing. However, solving for the competitive equilibrium is very easy.

- Derive individual supply
- Aggregate up to derive market supply
- Solve for price that clears the market, i.e. demand = supply.

Example 5.N: Suppose that $TC(q) = 0.1q^2 + 10$ Consider the following problem

- Derive individual supply equation.

- Suppose that there are 5 identical firms. Derive market supply equation.

Given that there are **two identical consumers** in the market with individual demand equation given by $P = 10 - q$. Solve for the market equilibrium.

How much is the profit that each firm earns in the equilibrium?

Long-run competitive equilibrium

- In the long-run, firms can freely enter or exit the industry.
    - If profit arises, outsiders will be attracted into the market.
    - If loss incurs, insiders will leave the industry

- Given that firms are driven by profit opportunity; the number of firm will be an endogenous variable.

- Long-run equilibrium would occur when each existing firm in the market receives zero profit. (no incentive to enter/exit!)

Definition:

Given the market demand, long-run competitive equilibrium comprises of allocation "Q^e", $\{q_1^e, q_2^e, \ldots, q_{N^e}^e\}$, "$N^e$", and "$P^e$" such that

(i) q_i^e is the profit-maximizing level of output at P^e.

(ii) $Q^e = q_1^e + q_2^e + \cdots + q_{N^e}^e$

(iii) P^e clears the market, i.e. $Q^e = demand$ at the equilibrium price

(iv) N^e results in zero-profit condition.

Algorithm:

- Each firm earns zero profit when each of them produce at the break-even price, i.e P = min(AC). At the break-even price, one also knows the break-even output for each firm.
- As a result, market equilibrium must be the break-even price.
- Given the break-even price, one can figure out total demand.

5.4.2) Monopoly market and equilibrium

- Single firm in the market "or" market with barriers to entry

- Monopolist firm takes control the whole market; firm can set price along the market demand curve.

- With the monopoly power, monopolist's revenue function is now given by

$$R(Q) = P(Q) * Q$$

- With the modified revenue function, the profit-maximizing conditioni is then turned into optimality condition implies the level of output such that

  - $MR(Q) = MC(Q)$

Example 5.p: Given market demand is $p = 10 - Q$ and $TC = \frac{1}{3}Q^3 - Q^2 + 6Q$ Find the level of profit-maximizing output, and price that producer would charge.

What do we know about the pricing policy of the monopoly?

- We can show that optimal behavior of monopoly can be alternatively characterized as the optimal mark-up (P/MC).

Example 5.0*:* Show that optimal behavior of the monopolist would entail them to choose for optimal mark-up where *the level of mark-up is inversely determined by elasticity of demand*.

Chapter 6: Function of several variables and multivariate calculus

6.1 Function of several variables

Suppose we have two variables (x, y). A function that maps these two variables into another set valued of variable, z, is called function of two variables.

Example 6.A $f(x,y) = 2x + x^2y^3$ $\quad z = x^{\frac{1}{2}} * y^{\frac{1}{2}}$

Similarly, a function that maps from the value of N-variable into another set valued of variable, z, is called function of N variables.

Example 6.B $f(x_1, x_2, \ldots x_n) = x_1x_2x_3 \ldots x_n$

Most economic functions require the representation of multivariate relationship.

Utility function: $U = f(x_1, x_2, x_3, \ldots, x_n)$

Production function: y = f(K,L,E); K = capital, L = Labor, E = Engergy

Demand function: $Q_x^d = f(P_x, I, P_y, T)$

Supply function: $Q_x^s = f(P_x, w, P_y)$

6.2 Multivariate differential calculus

6.2.1) Partial derivative: General concepts

Definition: Suppose $y = f(x_1, x_2, x_3, \dots, x_n)$, i.e. $f: R^n \to R,$ the partial derivative of "y" with respect to "x_i" is denoted by:

$$\frac{\partial y}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_i \quad = \lim_{h \to 0} \frac{f(x_1, \ \dots, \ x_i + h, \dots, x_n) - f(x_1, \ \dots, x_i, \dots, x_n)}{h}$$

- Operationally, deriving the partial derivative of a function is so simple!!
- We simply treat all other variables as constant, except x_i.
- Then, apply all the rule of differentiations that we knew from the case of single variable calculus to the multivariate function.
- Let's proceed to some examples.

Example 6.C:

Suppose that $U = -5x^2 - 12xy - 6y^2$ find $\frac{\partial U}{\partial x}$ $(= U_x)$ and $\frac{\partial U}{\partial y}$ $(= U_y)$.

Graphical illustration of partial derivative

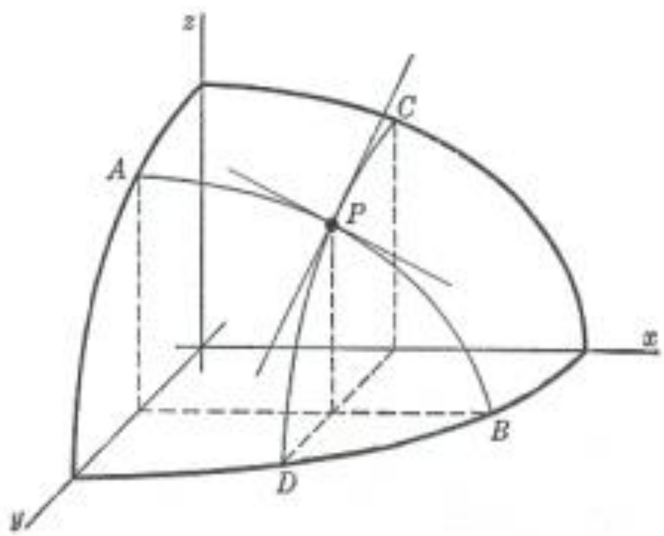


Fig. 1

Definition: *Gradient vector*

Vector of the Collection of all the first-order partial derivative terms in matrix (vector)

$$\nabla U = \begin{bmatrix} U_x \\ U_y \end{bmatrix}$$

For any arbitrary "n" variables, the gradient vector of $y = f(x_1, x_2, x_3, \dots, x_n)$ is given by

$$\nabla f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}_{n\times 1}$$

Example 6.D: $f(x_1, x_2, \dots x_n) = x_1 x_2 x_3 \dots x_n$

Higher-order partial derivative

- Second-order partial derivative: $\left(\frac{\partial^2 f}{\partial x_i^2}\right)$

Differentiating the first-order derivative function with the same argument: $\frac{\partial\left(\frac{\partial f}{\partial x_i}\right)}{\partial x_i}$, i.e. differentiating the multivariate function with respect to the ***same*** argument twice.

- Second-order cross partial derivative: $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) = \frac{\partial\left(\frac{\partial f}{\partial x_i}\right)}{\partial x_j} = f_{ij}$

What does the second-order partial derivative tell us?

- Concavity/Convexity along a certain axis!

Example 6.C ***(cont):*** What is the second-order partial derivatives of the function used in Example 6.C above

Definition:

Hessian matrix: Collection of the second-order partial derivative in matrix form

$$H = \begin{bmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{bmatrix}$$

Example 6.C (cont): Hessian matrix of U

Notice an important property of the Hessian matrix

For any arbitrary "n" variables, the gradient vector of $y = f(x_1, x_2, x_3, \ldots, x_n)$ is given by

$$H = \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{bmatrix}_{n\times n}$$

Example 6.D (cont): Hessian matrix of "f"

6.2.2) Partial derivative in the context of economics

6.2.1a) Marginal analysis

- Economically, what does the partial derivative, i. e. $\frac{\partial y}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_i \ (x_1, x_2, x_3, \dots, x_n)$, tell us about?
- Marginal effect of x_i on y. (marginal treatments)
    - Measuring the change in "y" that is purely attributed to the change in particular "x", while keeping all other "x" stay the same.
- This basically captures the concept so called "*Ceteris Paribus*".

Example 6.E Consider a production function given by, $Q = K^{\frac{1}{3}}L^{\frac{2}{3}}$

a) Derive the expression for marginal product of labor (MPL) and capital (MPK), respectively

b) How does the MPK change with respect to k? Similarly, how does the MPL change with respect to L?

c) How does the MPL change with respect to k? Discuss the implication of you result.

Cobb-Douglass function

- Charles W. Cobb and Paul H. Douglas (1928) *"A Theory of Production"* AER: cited as one of the top 20 most influential paper in economics
- Functional form: $Q = K^{\alpha} L^{\beta}$;
  - In our example 6.E, $\alpha = \frac{1}{3}$ and $\beta = \frac{2}{3}$
- Mathematically, the Cobb- Douglass production function is a ***homogenous function***.
  - $\alpha + \beta$ is the degree of homogeneity.

__Definition:__ A function is said to be a homogenous function if

$$f(tx_1, tx_2, \dots, tx_n) = t^m f(x_1, x_2, \dots, x_n)$$

where "m" is called the degree of homogeneity.

The Cobb-Douglass function is a homogenous function

- **Proof:** $(tK)^{\alpha}(tL)^{\beta} = t^{\alpha+\beta} K^{\alpha} L^{\beta} = t^{\alpha+\beta} Q$

Example 6.F: Which one is HM function, and to what degree.

- $z = x^2 + y^2$
- $z = x + y^2$
- $z = x^2 y + x^{\frac{8}{3}} y^{\frac{1}{3}}$
- $z = (\alpha K^{1-a} + \beta L^{1-a})^{\frac{1}{1-a}}$

Important properties of Homogenous function

(i) f_i is degree of homogeneity "m-1".

(ii) Euler theorem: $\sum_{i=1}^{N} f_i x_i = mf(x_1, x_2, \dots, x_N)$

Why do we use HM function in economics?

- For the production, HM reflects the assumption that production technology exhibits return to scale.
- What is the return to scale?
  - If you *proportionately* increase "K" and "L", how much is the increase in "Q", measuring in terms of its proportion to an increase in factor input.
- Degree of homogeneity can tell us:

$$\alpha + \beta = 1: \text{ Constant return to scale}$$

$$\alpha + \beta > 1: \text{ Increasing return to scale}$$

$$\alpha + \beta < 1: \text{ Decreasing return to scale}$$

6.2.1b) Elasticity versus Partial elasticity

- $y = f(x)$ ➔ Elasticity $= \frac{dy}{dx} * \frac{x}{y} = \frac{d\ln(y)}{d\ln(x)}$
- $z = f(w, x)$
  - Partial elasticity of z with respect on w

$$= \frac{\partial z}{\partial w} * \frac{w}{z} = \frac{\partial \ln(z)}{\partial \ln(w)}$$

  - Partial elasticity of z with respect on x

$$= \frac{\partial z}{\partial x} * \frac{x}{z} = \frac{\partial \ln(z)}{\partial \ln(x)}$$

Example 6.F: Consider a demand function with the form: $Q = Ap^{-0.28}m^{-0.34}$, where $p = price \;\; and \;\; m = income$.

6.2.1c) Comparative Static Analysis using partial derivative

- Model → solve for reduced-form equations of endogenous variables.
    - Endo as a function of (i) exogenous variables and (ii) parameters
    - We can apply partial derivate to the derived endogenous equilibrium solution.
    - This allows us to see how endogenous variables respond to the changes in exogenous variables and parameters.

Example 6. G: Recall that in a simple macroeconomic model, the solution for equilibrium output can be given by $y^* = \frac{I_0+G_0}{1-(1-t)b}$, where t is level of income tax and b is marginal propensity to consume. What about the effect of an increase in "b" and an increase in "t" on output?

6.2.3) Total differential

- Total differential in "y" = Total change in "y"
- Recall the concept of single variable differential
    - $dy = f^{'}(x)dx$
    - $f^{'}(x) = marginal\ change\ in$ "y" $per\ a\ unit\ of\ change\ in\ x.$
    - dx = units of change in "x".
- How about total differential in multivariate relationship.

Definition: Suppose $y = f(x_1, x_2, x_3, \dots, x_n)$

$$dy = \sum_{i=1}^{N} f_i dx_i$$

Tips: *When you are taking the total differential, you are just taking all the partial derivatives and adding them up.*

Example 6.H: Suppose $(y, r) = \sqrt{\frac{y}{r}}$. Derive total differential of C.

Example 6.J: Consider the equilibrium function of real income $y^* = f(I_0, G_o, t, b) = \frac{I_0+G_0}{1-(1-t)b}$. Find total differential of y^*.

6.2.4) Total derivative (aka. Chain rule!)

- For an illustrative purpose, consider the consumption function in **example 6.H.** Suppose that *"y" and "r" grows over time.*
- That is, y = g(t) and r = h(t) where t is the numbers of period from now.
- By the composite function, consumption will be *ultimately* determined by "t".
- Then, how do we calculate the derivative of "c" with respect to "t"?
- This requires the concept of total derivative.

- Total differential in "c" can be given by

$$dC = \frac{\partial c}{\partial y}dy + \frac{\partial c}{\partial r}dr$$

- Given this assumption, one obtains that

$$dy = g^{'}(t)dt$$

$$dr = h^{'}(t)dt$$

- Applying the two expressions above, we obtain that

$$dC = \frac{\partial c}{\partial y}g^{'}(t)dt + \frac{\partial c}{\partial r}h^{'}(t)dt$$

- As a result, total derivative of C with respect to t is given by

$$\frac{dC}{dt} = \frac{\partial c}{\partial y}g^{'}(t) + \frac{\partial c}{\partial r}h^{'}(t)$$

Example 6.K Consider a consumption function with $C = t^2 + \sqrt{y} - r^3$ and $y = 2t^3$ and $r = t + 1.$ Derive the total derivative of C with respect to t. Calculate the value of dc when t = 0

6.2.5) Partial total derivative (aka. Chain rule for multivariate!)

- From the example above, what if both y and r depend on "t" and "s"?
- Consumption would then be ultimately driven by "t" and "s".
- The concept is changed to *partial total derivative*: $\frac{\partial c}{\partial t}$ and $\frac{\partial c}{\partial s}$

Following the same method as we applied to total derivative, we know that

$$dC = \frac{\partial c}{\partial y}\left(\frac{\partial y}{\partial t}dt + \frac{\partial y}{\partial s}ds\right) + \frac{\partial c}{\partial r}\left(\frac{\partial r}{\partial t}dt + \frac{\partial r}{\partial s}ds\right)$$

As a result,

$$\frac{\partial c}{\partial t} = \frac{\partial c}{\partial y}\left(\frac{\partial y}{\partial t}\right) + \frac{\partial c}{\partial r}\left(\frac{\partial r}{\partial t}\right)$$

Meanwhile, we would yield that

$$\frac{\partial c}{\partial s} = \frac{\partial c}{\partial y}\left(\frac{\partial y}{\partial s}\right) + \frac{\partial c}{\partial r}\left(\frac{\partial r}{\partial s}\right)$$

6.3 The Implicit derivative

6.3.1) Implicit derivative: Genearl concepts

- Derivative method applied to an ***implicit function***
- Let's first dsicuss what implicit functions are.

Explicit function	**Implicit function**
• $y = 2x - 1$ • $y = x^2 + 2x + 1$ • $y = \ln(2x+1) + 2x + 1$ • $z = \ln(2x+1) + 2y^2x$	• $x^2 + y^2 = 3$ • $xw + yw - w^2 + zx^2 = 9$

For an implicit function, how do we obtain the derivative of y with respect to x?

Method 1: (brute force) → Rewrite y in terms of x.

- Comments:
  - Tedious... and NOT applicable for all the cases.
  - In many cases, the function doesn't admit any closed-form solutions: impossible to rewrite "y" in term of "x".

Example 6.L Suppose that $x^2 + y^2 = 3$, find dy/dx.

Method 2: Apply the implicit function theorem

Theorem: Suppose a given function is written in the form of $F(x, y) = 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Proof: Use the totoal differntials

Example 6.M Suppose that $\ln(x+2) - y^2 - e^{x+y} = 3$, *Find* $\frac{dy}{dx}$

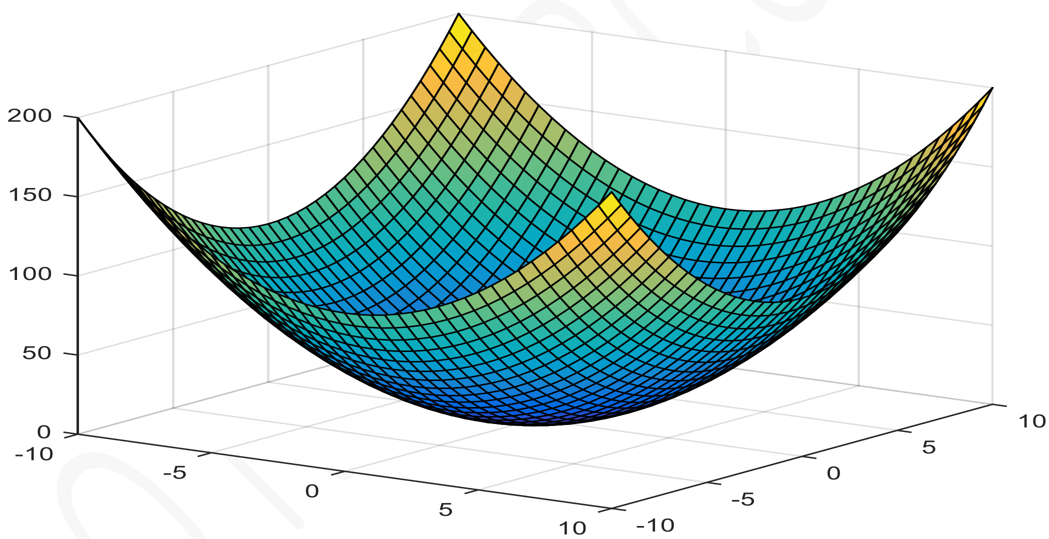
6.3.2) Implicit derivative and Level set (Contour set)

Definition: Suppose a function $z = f(x, y)$, the level set of "f", denoted by $L(f, z_0)$, is defined as

$$L(f, z_0) = \{(x, y) | \; f(x, y) = z_0\}$$

All possible combinations of (x,y) that generate f(x,y) = z_0.

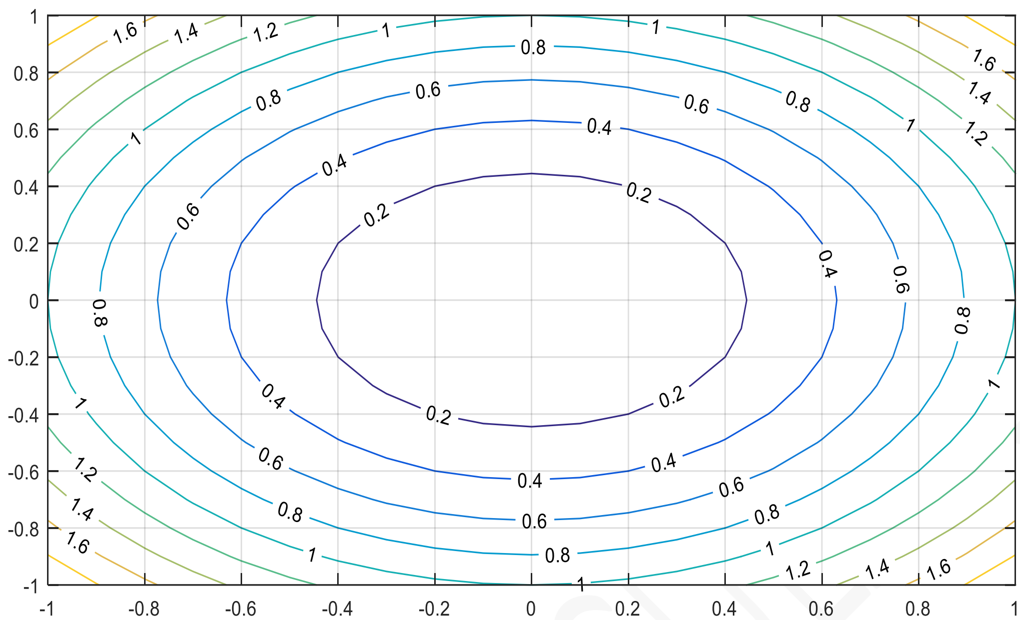
- For example, $z = f(x, y) = x^2 + y^2$



Assuming that $z_0 = 1$, $\quad L(f, 1\,) = \{(x, y)|x^2 + y^2 = 1\}$

Graphically, $L(f, 1\,)$ can be represented in a 2-D figure. The shape of the relationship is a circle with unit radius.

Contour of level sets: By varying the value of "z", we yield several contour of level sets.



Derivative of the level set

- One can apply the second result of the implicit function theorem.

$$\frac{dy}{dx}|_{z=z_0} = -\frac{x}{y}, \qquad i.e.\ \ dz = 0$$

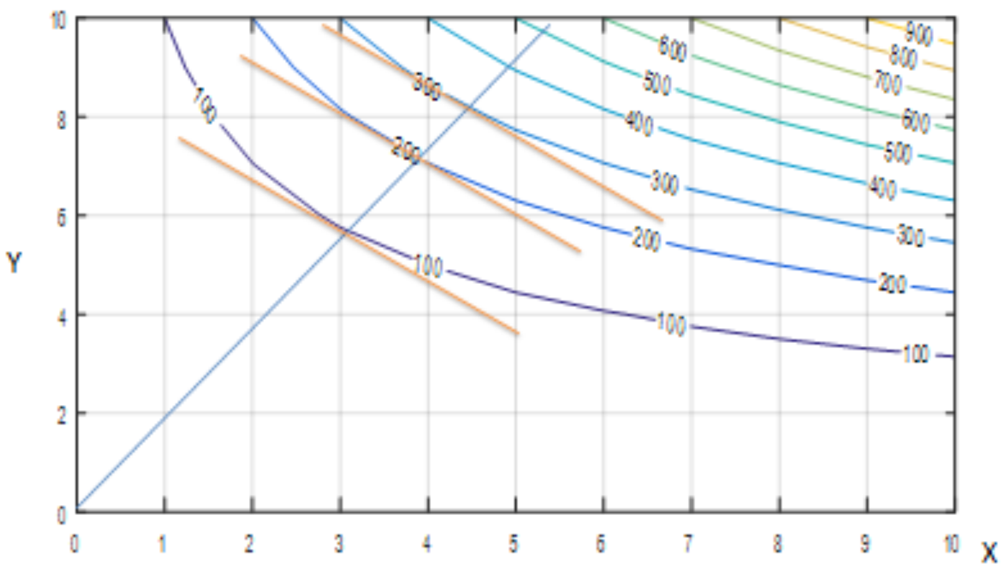
Economic example of the level set

Example 6.O: Suppose that U is the utility function of a consumer. The function takes the following form: $U = U(x, y) = xy^2$ where x is the amount of consumption on good "x" and y is the amount of consumption on good "y".

a) Derive the level set of the utility function. What do we call the level set of utility function in economics?

b) Calculate slope of the level set. What does the slope mean to you in economics?

Locus of indifference curves



30

6.3.3) Implicit function and its application for comparative static analysis

6.3.1a) Comparative static analysis without explicit solution

Motivating example: Consider a market model where market demand and market supply are given by,

$$Q^d = D(P, y) = a - bP + cy,$$

$$Q^s = S(P) = e + fp$$

Comparative static analysis: ***how does the level of income affect equilibrium output and price?***

Approach for the analysis:

- Mechanically, we solve for the solution (reduced-form solution)
- Apply derivative technique, and check for the sign of derivative.
- Then reach the conclusion over the impact of income (exogenous changes) on equilibrium output and price (the two endogenous variables in our model.)

These are the procedure that we did for the midterm exam. Then, what's new?

Motivating question: would the conclusion be obtained under a wider class of the model equations? (e.g. non-linear function) Could it be shown mathematically? $\frac{\partial Q^*}{\partial y}, \ \frac{\partial P^*}{\partial y} = ???$

Why is this important? → Economics usually want to get some testable implications under more ***robust*** assumptions.

- People might find it less convincing to assume linear demand, and hence arguing that your conclusion is only true under the too restricted assumptions.
- But, people might be more comfortable with the assumption that law of demand and law of supply holds ($D_p < 0$; $S_p < 0$), together with assuming that the production in question is normal goods ($D_y > 0$).

Implicit derivative can be applied into this context. How?

Note first:

$$S(P) = D(P, y)$$

Under some conditions, we should be able to write P in terms of Y as the explicit function, i.e. deriving the reduced-form equation.

6.3.1b) Comparative static in an aspect of optimization

Consider a profit- maximization problem $\pi(L) = pf(L) - wL$ where $f(L)$ is increasing and concave in L.

Following the first-order condition:

By applying the implicit function, we yield that

6.3.1c) Comparative static by the Implicit function theorem: simultaneous approach.

Consider a simple macroeconomics with three equations

Market clearing condition: $Y = C + I$

Consumption function: $C = f(Y;r) \qquad 0 < f_y(Y;r) < 1 \;\; and \;\; f_r(Y;r) \leq 0$

Investment function: $I = g(Y;r,I_0)$

$$0 < g_y(Y;r,I_0) < 1, \;\; g_r(Y;r,I_0) \leq 0 \; and \;\; g_{I_0}(Y;r,I_0) > 0$$

In this model, we are now treating "r" and "I_0" as the two exogenous variables.

By solving for the solution of simultaneous equations, we know that, ***under some regular conditions***, the reduced-form equations must exist,

$$Y^* = Y^*(r,I_0); \qquad C^* = C^*(r,I_0); \qquad I^* = I^*(r,I_0)$$

How can we perform the sensitivity analysis, based on unknown functional form of the reduced-form equation?

First, we know that the three reduced-form equation must satisfy for the three original equations above.

Market clearing condition: $\boldsymbol{Y^*(r,I_0) = C^*(r,I_0) + I^*(r,I_0)}$

Consumption function: $\boldsymbol{C^*(r,I_0) = f(Y^*(r,I_0);r)}$

Investment function: $\boldsymbol{I^*(r,I_0) = g(Y^*(r,I_0);r,I_0)}$

Second, the equilibrium relationship must also hold for the *differential* level.

Market clearing condition: $Y = C + I$

Consumption function: $C = f(Y;r) \qquad 0 < f_y(Y;r) < 1 \;\; and \;\; f_r(Y;r) \leq 0$

Investment function: $I = g(Y;r,I_0)$

Chapter 7: Mathematical optimization: Unconstrained problem

7.1 General Statement of the problem

Two classes of the optimization problem

(i) Unconstraint optimization

$$\max_{x_1, x_2, \ldots, x_N} (\min) f(x_1, x_2, \ldots, x_N)$$

Comments: You are free to choose combinations of "x".

Example:

(ii) *Constraint optimization*

$$\max_{x_1, x_2, \ldots, x_N} f(x_1, x_2, \ldots, x_N)$$

such that $g(x_1, x_2, \ldots, x_N) = b$

Comments:

a) *M-constraint* (equations) functions

b) Constraint set could be described/represented by ***inequality***,

$x + y = 2 \;\; v.s. \;\; x + y \leq 2$ → Line V.S. Area shape

This topic is the central discussion for EE421.

7.2 Unconstrained optimization

- This section develops toolkits to solve for the solution of the unconstraint optimization problem

7.2.1) Types of optimizer revisited!

Definition: Local optimizer

(x^*, y^*) is a ***local*** optimizer of $f(x, y)$ if and only if, for all (x', y') that are closed to (x^*, y^*),

$f(x^*, y^*) \geq f(x', y')$ (local max)

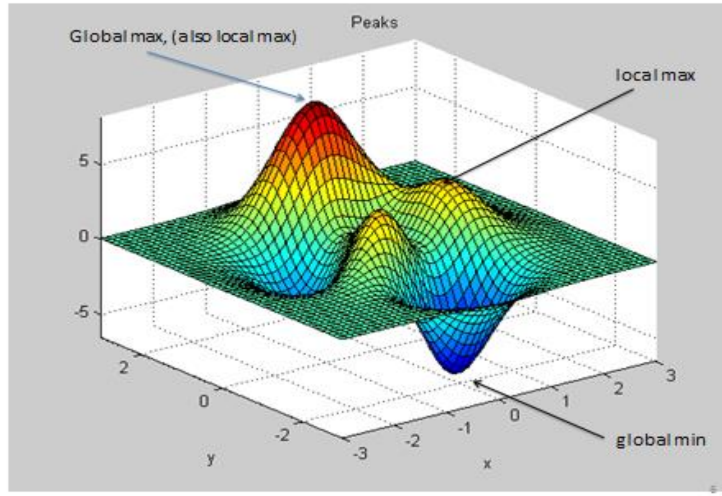
$f(x^*, y^*) \leq f(x', y')$ (local min)

How do mathematicians define the closed-to set?

- x^* ➔ x' in the neighborhood of x^*
  - $(x^* - \varepsilon, x^* + \varepsilon)$
  - Distant between x' and x^* is less than ε.

- (x^*, y^*) ➔ (x', y') in the neighborhood of (x^*, y^*)
  - Distant between (x', y') and (x^*, y^*) is less than ε.
  - $\forall (x', y')$: $\sqrt{(x' - x^*)^2 + (y' - y^*)^2} < \varepsilon$

Definition: Global optimizer



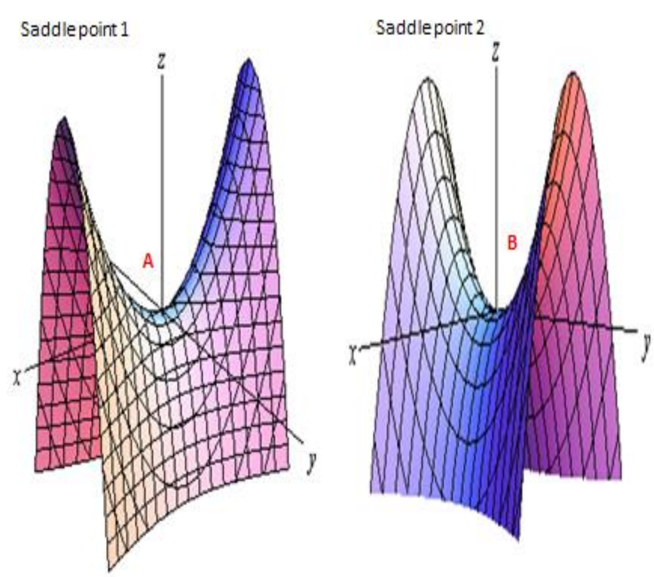
(x^*, y^*) is a ***global*** optimizer of $f(x,y)$ if and only if, for all (x', y') in the **entire domain set** that defines "f",

$f(x^*, y^*) \geq f(x', y')$ (global max)

$f(x^*, y^*) \leq f(x', y')$ (global min)

- **Comments**:
  - If the point is local max, it must be local max on both axes (directions).
  - If the point is local max on one axis (direction), but local min on the other axis (direction), the point is neither considered as local max nor local min. Instead, the point is called "***saddle point***".

Definition: Saddle point 1



(x^*, y^*) is a *saddle point* of $f(x,y)$ if and only if, for all (x', y') that are closed to (x^*, y^*),

$f(x^*, y^*) \geq f(x^*, y')$ (max in y)

$f(x^*, y^*) \leq f(x', y^*)$ (min in x)

Definition: Saddle point 2

(x^*, y^*) is a *saddle point* of $f(x,y)$ if and only if, for all (x', y') that are closed to (x^*, y^*),

$f(x^*, y^*) \leq f(x^*, y')$ (min in y)

$f(x^*, y^*) \geq f(x', y^*)$ (max in x)

7.2.2) Unconstraint (constraint-free) optimization: theorem

- We start out with a 2-varaible case. Later, we will generalize the result into N-variable case.

7.2.1a) Two-variable choice problem

Theorem: *Properties of Local optimizer*

Suppose that $f(x,y)$ is a twice-differentiable function. That is, both ∇f and Hessian matrix exist. We would observe the following two properties for which the local optimizer (x^*,y^*) must hold.

- $\nabla f\ (x^*,y^*\ ) = 0$. (**First-order necessary condition**)
- If (x^*,y^*) is the local maximizer (minimizer) of $f(x,y)$, then $f(x,y)$ is concave (convex) at (x^*,y^*). (**Second-order necessary condition**)

Theorem: *Necessary and Sufficient condition*

This theorem is commonly known as the two-step cooked procedure for locating local optimizers. Given that f(x,y) is twice differentiable function,

(i) **First-order condition:** Locate (x^*,y^*) such that Gradient of the function is equal to zero.

(ii) **Second-order condition:** Distinguishing the type of local optimizers by checking for the type of curvature at the point located from (i).
  a. Concave → local maximizer
  b. Convex → local minimizer

The question is "How do we check for the concavity/convexity of a function"?

Concavity/convexity of a function

In single variable case, we check for the concavity/convexity of a function using the second-order derivative.

- This is fine, because we only have one axis/direction of change to consider.

For the multivariate case, it is **no longer** appropriate to simply look at the second-order partial derivative. Mathematically, we check for the concavity/convexity of a multivariate function using the **second-order total differential**.

- $f(x, y)$ is said to be *convex* at the point (x^*, y^*) if and only if

$$d^2 f(x^*, y^*) \geq 0$$

- $f(x, y)$ is said to be *concave* at the point (x^*, y^*) if and only if

$$d^2 f(x^*, y^*) \leq 0$$

Example 7.A Derive the expression for the second-order differential of f(x,y)

Theorem:

- If the Hessian matrix is *positive definite*, then $d^2f(x^*, y^*) > 0$ for all possible combinations of dx and dy.
    - Implies that "f" is convex in the neighborhood of (x^*, y^*)!

- If Hessian matrix is *negative definite*, then $d^2f(x^*, y^*) < 0$ for all possible combinations of dx and dy.
    - Implies that "f" is concave in the neighborhood of (x^*, y^*)!

Theorem:

Suppose that A is a ***symmetric*** matrix.

- A is positive definite if the determinants of all the ***leading principal minors*** are positive.
- A is negative definite if the determinants of all the ***leading principal minors*** have alternate signs, *with* the negative value for the determinant of the initial leading principal minor A_1.

Comments: Since "H" is symmetric, we can apply the theorem to check for the definiteness of "H" using the above theorem. An important question remained is the definition of what so called leading principle minors.

Definition: Leading principal minors

- All the sub-matrices of "A", indexed by A_i, each of which is expanded along the diagonal of the original matrix A.

$$A = \begin{bmatrix} 1 & -4 & 3 \\ -4 & 5 & 0 \\ 3 & 0 & 9 \end{bmatrix} \Rightarrow A_1 = [1]; \quad A_2 = \begin{bmatrix} 1 & -4 \\ -4 & 5 \end{bmatrix}; \quad \mathrm{A}_3 = A$$

Note: There exists "N" leading principal minors for an N-by-N matrix.

Example 7.B: $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$; $A = \begin{bmatrix} -1 & 2 \\ 2 & -5 \end{bmatrix}$

Find all the sub-matrices of A and check for definiteness of A.

From the theorems stated above, we can conclude that, *under the 2-variable case*, second-order condition requires that

- For the minimizer, H is positive definite = "f" is convex
  - $\det(H_1) = f_{xx} > 0$
  - $\det(H_2) \ = f_{xx} * f_{yy} - f_{xy}^2 > 0$

- For the maximizer, H is negative definite = "f" is concave
  - $\det(H_1) = f_{xx} < 0$
  - $\det(H_2) \ = f_{xx} * f_{yy} - f_{xy}^2 > 0$

Conclusion:

Example 7.C: Determine all the optimizers of

$$f(x,y) = x^3 - y^3 - 9xy$$

Example 7.D: Determine all the optimizers of

$$f(x,y) = x^2 - 2xy + 3y^2 + 2x - 2y$$

Definition: *Globally concave/convex*

A function is said to be globally concave (convex) if the function is concave (convex) for the *entire domain set* defining the function.

Following the theorem of definiteness on matrix, we can check for the global property of a function.

Lemma:

- Globally concave:
  - Hessian matrix is negative definite *for all (x,y)*.
- Globally convex
  - Hessian matrix is positive definite *for all (x,y)*.

Example: $\boldsymbol{f(x,y) = e^x + e^y}$, $\quad \boldsymbol{g(x,y) = -(x^2 + y^2)}$

Why do we care about this?

- Optimized solution in economics attempt at searching for the first-best, i.e. global solution.
- Unfortunately, mathematical optimization only provides toolkits for searching/identifying local optimizers.

Theorem:

If a function is globally convex (concave), local minimizer (maximizer) is warranted to be global minimizer (maximizer).

7.2.2b) N-variable case

Now we consider the generalized version of the unconstraint optimization with N variables.

Objective function: $z = f(x_1, x_2, x_3, \ldots, x_n)$

The solution method is similar; we basically extend the two-step procedues as outlined before. Previously, we have two equations for the first-order condition. When the problem is extended into N variable, the set of equations that satisfies the first-order conditions will be similarly extended into N equations.

FOC.

$f_1 = f_2 = f_3 = \cdots = f_n = 0$ ➔ solve for the solution from FOCs.

The second-order derivative matrix will be a square matrix with N x N. One need to check for its definiteness, using the theorem outlined before. IN this case, there will be N sub-matrices that one need to check for their sign.

SOC.

$$H = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix},$$

• H is negative definite ➔ ○ $\lvert H_1\rvert < 0, \lvert H_2\rvert > 0, \lvert H_3\rvert < 0$ …. ○ sign($\lvert H_i\rvert$) = $(-1)^i$ for each "i". ○ Local min	• H is positive definite ➔ ○ $\lvert H_1\rvert > 0, \lvert H_2\rvert > 0, \lvert H_3\rvert > 0$ …. ○ sign($\lvert H_i\rvert$) > 0 for *all* "i". ○ Local max

7.2.3) Some economics applications

7.2.3a) Basic profit maximization and Factor inputs decision

- **Structure of the problem**
    a. Firm owns a technology that requires the use of some types of factor inputs.
    b. Technology can be represented by a production function.
        i. For simplicity, I will assume only two types of the factor inputs, capital (K) and labor (L)
        ii. Capital costs firm "r" per unit of installation, and labor costs firm "w" per unit of labor usage.
    c. Firm sells the product to the market, getting back "p" in return, per unit of output sold.
    d. Question under the problem is that "*how much to utilize each type of factor input to get the highest level of profit?*"

Objective function:

Choice variables:

Optimality conditions: FOC

Second-order condition

- Hessian of profit function is

$$H = \begin{bmatrix} \pi_{KK} & \pi_{KL} \\ \pi_{LK} & \pi_{LL} \end{bmatrix} = \begin{bmatrix} pf_{KK} & pf_{KL} \\ pf_{LK} & pf_{LL} \end{bmatrix} = p \begin{bmatrix} f_{KK} & f_{KL} \\ f_{LK} & f_{LL} \end{bmatrix}$$

- So, we need $\begin{bmatrix} f_{KK} & f_{KL} \\ f_{LK} & f_{LL} \end{bmatrix}$ to be negative definite because $p > 0$.
    - This condition ensures the solution from FOC as a maximizer.
    - What does it mean in economics for having negative definitive in $\begin{bmatrix} f_{KK} & f_{KL} \\ f_{LK} & f_{LL} \end{bmatrix}$?
    - What could it be implied about the mathematical property of the production function?

Example 7.E Suppose $Q = K^{\frac{1}{2}} + L^{\frac{1}{2}}$ and price is fixed, equal to P. Assume that "w" as wages and "r" as rental.

a) Setting up the profit function.

b) Derive the optimal level of capital installation and labor usage.

c) Check the second-order condition to confirm that your answer is correct.

d) Calculate the level of maximized (optimal) profit.

e) What is the optimal level of capital installation and labor usage when P = 1000, w = 20 and r = 10.

Exercise 7.A

Redo the **example 7.E**, but using the following Cobb-Douglas function: $\mathrm{Q} = K^2 L^3$ as the production technology of the firm.

- For any aribitary level of "w", "r" and "p", derive the demand function for both capital and labor.
- Use the partial derivative, discuss some important properties of demand function of capital and labor.
- Derive the optimal profit function, and discuss some properties of the function.

7.2.3b) Multi-plant problem

Structure of the problem:

- Firm operates in a market, and sells only a single type of product in the market.

- Firm has several plants that can be used to fulfill the production. (Think about plants that are located in different cities, for example. Or, two plants in the same location, but each differs in their production technique.)

- How to optimality choose the capacity of each plant, given a certain level of the total output that firm attempts to sell to the market.

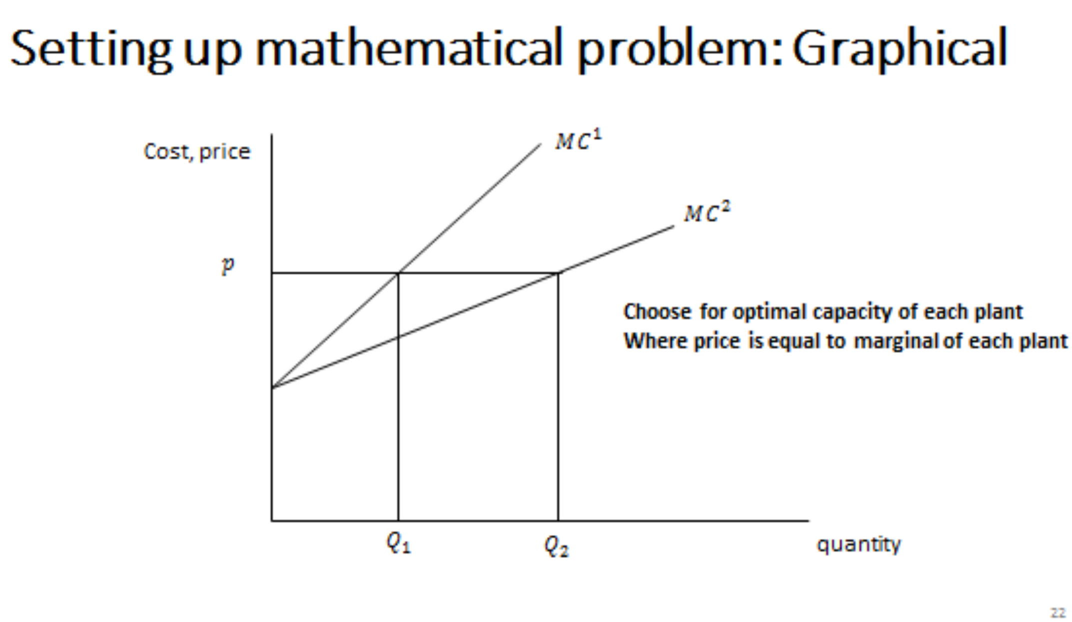
For simplicity, consider a 2-plant example.

- $C^1 = C^1(Q_1) = cost\ of\ plant\#\ 1,\ when\ firm\ chooses\ to\ produce\ Q_1$

- $C^2 = C^2(Q_2) = cost\ of\ plant\#\ 2,\ when\ firm\ chooses\ to\ produce\ Q_1$

The total production of these two plants must fulfil the total output (Q) that firm is trying to sell to the market. That is, $Q_1 + Q_2 = Q$

First, consider a simple case where firm is a price taker. That is, it treats price (p) as given. Given the situation, the optimal plant capacity that maximizes overall profit can be determined using the first-order condition,



Example 7.F Suppose that P = 25 and the cost function for each plant is given by,

$$C^1(Q_1) = 2Q_1^2 + 5Q_1 + 10$$

$$C^2(Q_2) = 2Q_2^2 + 3Q_2 + 15$$

Q_1^* *and* Q_2^* that maximizes the profit?

Example 7.G Multi-plant problem of a monopolist.

Suppose a monopolist faces the market demand given by: $P = 10 - Q$

$$TC(Q_1) = 2Q_1^2 + 5Q_1 + 10$$

$$TC(Q_2) = 2Q_2^2 + 3Q_2 + 15$$

Determine Q_1^* *and* Q_2^* that maximizes the profit.

7.2.3c) Multi-product problem

Structure of the problem

- Firm produces different types of products, and distributes those products to different markets.

- Each product can be sold at different prices.

    - In each market, price might be given, or can be set if the firm has market power.

    - Start from the case with given price, first.

Example 7.H: Suppose the total cost function of a perfectly competitive firm is given by $\mathrm{TC}(\mathrm{Q}_1, \mathrm{Q}_2) = 2Q_1^2 + Q_1 Q_2 + Q_2^2$. Determine the optimal level of $(\mathrm{Q}_1, \mathrm{Q}_2)$.

Example 7.I Suppose $P_1 = 35 - Q_1$ and $P_2 = 33 - Q_2$. Determine the optimal level of output for the two types of product if monopolist's cost function can be given by, $C = 2\mathrm{Q}_1^2 + \mathrm{Q}_1\mathrm{Q}_2 + 2\mathrm{Q}_2^2$.

7.2.3d) Price discrimination: Third-degree price discrimination

Conventional set-up for the single-product problem a monopolist is given as follows

- Suppose the monopolist faces a downward sloping demand curve, given by: $P = p(Q)$. And the cost function is given by $C(Q)$.
- What is the optimal policy for the monopolist?
  - Choose Q so that monopolist's profit is maximized.
  - Requires setting Q such that: $MR(Q) = MC\,(Q)$.
- Recall that $MR(Q) = P\left(1 - \frac{1}{|\varepsilon_d|}\right)$.
- Given this, we know that:

$$P\left(1 - \frac{1}{|\varepsilon_d|}\right) = MC$$

$$\frac{P}{MC} = \frac{|\varepsilon_d|}{|\varepsilon_d| - 1}$$

- Two implications for pricing setting of the monopolist.

(i) Monopolist will **NOT** produce in the region that demand is inelastic.

$$\frac{P}{MC} = \frac{|\varepsilon_d|}{|\varepsilon_d| - 1}$$

(ii) Having produced under elastic region, $\frac{P}{MC} > 1$.

- Optimal price setting entails firm to set their price above the marginal cost, i.e. setting "mark-up" optimally.
- The optimal mark-up pricing inversely related the elasticity of demand.
- The more elastic demand monopolist faces, the lower degree of mark-up.

Comments:

- Simple, single pricing.
    - In the industrial organization (EE481), this is called "single *linear-pricing*".
    - Pricing is linear in the sense that price-per-unit is set the same across different consumers.
    - While it's true that the more you buy, the more you pay in totals. However, each unit of your purchase, price-per-unit remains the same.
- **Is there any pricing strategy that could improve the outcome?**
    - Yes, by the price discrimination strategy.

What is the price discrimination?

- Any pricing strategy aimed at further increasing the profit, i.e. to higher level of profit than that could be attained under single linear-pricing method.
- Might be possible because buyers in the market are actually *heterogeneous*.
    - Rich/poor... High-demand/ low-demand... variety of choices/limited choices
- So, pricing scheme that is based on some socio-economic conditions might be helpful. (That's why it is called "*discriminatory*".)

First-degree price discrimination

- The strongest version of the price discrimination strategy.
- Varying price for every single units of the purchase.

Second-degree price discrimination

Third-degree price discrimination

- The two versions of the price discrimination discussed above require great deal of information, and therefore it is costly for the practical implementation.

- The third-degree price discrimination proposes a comprised solution in which it is easier to implement.

- That is, monopolist charges for N linear pricing with respect to N different types of consumers. (multiples linear pricing)

- While there is a continuum of buyers in the market, it's possible that we can group those buyers into N groups, each has shared some similar characteristics in economics aspect.

Structure of the problem:

- A single monopolist firm selling its product to N groups of buyers; for simplicity, let's assume two: namely A and B.

  Group A: $\boldsymbol{P_A = D_A(Q_A)}$

  Group B: $\boldsymbol{P_B = D_B(Q_B)}$

  - Note: each group MUST differ in their characteristics and some socio-economic profiles. → to be explained later!

- The product costs the same across different types of consumers.

  - It's not price discrimination if the product costs differently across types of consumers. (You don't know what in fact causes the price to be high in one, but low in the other market. It could be attributed to cost differentials.)

    Cost: $\boldsymbol{C = C(Q) = C(Q_A + Q_B)}$

Example: Consider a plastic producer that produces plastic and sells its product to two different types of users. The first type uses the plastic in dental clinics. The second type uses the plastic as inputs for manufacturing some other products. Let's call the first group as "D"; meanwhile, the second group as "M"

$$P_D = 300 - Q_D$$

$$P_M = 100 - \frac{1}{2}Q_M$$

Suppose further that marginal cost is "$2" per a unit of production.

Find the profit-maximizing output under the third-degree price discrimination.

Find the profit-maximizing output when the third-degree price discrimination is impractical.

7.2.3e) Cournot model

- So far, we have discussed the only two *extreme* cases of market structure under which a representative firm operates: *perfect competitive and monopolist.*
- In many industries, the market has more than one firm, but not too many. Firm's behavior, and hence equilibrium outcome, in those markets are different from the theoretical predictions that we obtain from the model built upon those two extreme assumptions.
- Micro-theorists have developed some models that can be used to understand and predict the behavior of firms under the situation when there are few numbers of firms operating.

- This is known as the ***Cournot model***.

- What the Cournot model captures:

  - Prediction of equilibrium outcome of market in which there are few numbers of firms.

    - Few enough that each firm has to make their decision based on some strategic interactions.

    - A situation is called *strategic interaction* if the outcome/pay-off that a person would receive depends upon the decision/action taken by others, or vice versa.
      - Chess game
      - B&S: battle of the sex

  - What is the equilibrium outcome?
    - Firm's Optimality
    - Market clearing or self-sustaining action.

Example: Suppose the 2 firms in the market: namely firm A and firm B. Each firm faces the same market demand: $P = 50 - 2Q$ where Q is total output in the market, P is price per unit. What is the optimal choice of output for firm A when the cost function is given by $C_A = 2Q_A$?

The condition obtained from the above example is generally known under several names, ranging from "*optimal reaction function*", "*optimal contigent plan*", or "*best respose function*".

Definition: Best response function

Let (a_1, a_2) be action profiles that players can choose in a game. Define $u^1(a_1, a_2)\ and\ u^2(a_1, a_2)$ as the pay-off that arise from having the two players chosen the profile (a_1, a_2). The best response function of each player, indexed by i, is the contingent plan that prescribes the decision rule of its own, based on all possible actions that could be chosen by the other players.

$$BR^1 = a_1^* \in \max_{a_1} u(a_1, a_2) ==> a_1 = BR^1(a_2)$$

$$BR^2 = a_2^* \in \max_{a_2} u(a_1, a_2) ==> a_2 = BR^2(a_1)$$

Example: Derive the best response function of firm B, when the cost function of firm B is given by $C_B = 2Q_B$.

Solution concept under strategic environment

Notice that each firm strategically thinks in a similar fashion. They solve for the "*contigengy plan*". To solve for the equilibrium solution under this strategic environment, one generally resorts their idea to the concept of *self-sustating situation*. To understand this, let's put the two best responses functions in the same figure, and interprete what they are.

Example: Best response functions *and* unsustainable outcomes

The equilibrium prediction under the strategic interaction is well studied by game theorist. John Nash has long developed the notation of so called *Nash Equilibrium*, one of the common equilbirum notion widely used in economics and game theory.

Definition: Nash-equilibrium

(a_1^N, a_2^N) is said to be a Nash equilibrium if

(i) $u^1(a_1^N, a_2^N) \geq u^1(a_1\ , a_2^N)$

(ii) $u^2(a_1^N, a_2^N) \geq u^2(a_1^N, a_2\ )$

That is, no one is deviating from the proposed solution.

Result: Nash equilibrium profile can be obtained by solving for the solution of simultaneous equations given by best responses functions.

Example: Cournot equilibrium; quantity, price and profits

An equilibrium with Merger

Let's consider an alternative arrangement in which both firms agree to coorindate and act as a monopolist. So, rather than acting as a self-interest maximizing firm, each firm acts to maximize group's benefit. The situation captures the concept of collusion or cartel. Should they collude?

7.3 Constrained optimization

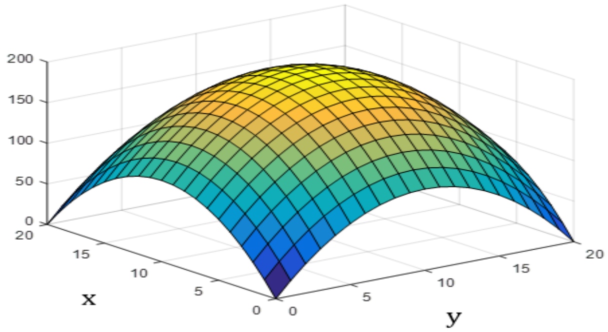
Most economics problems are best described by constrained optimization; in general, one cannot freely choose what he/she wants. Generally, one must face with a trade-off situation; that's why we study economics. This section discusses about the techniques applied to the constrained optimization problem. We first start with a 2-variable case and single constraint, and then discuss the more general version where there are *n variables* with single constraint.

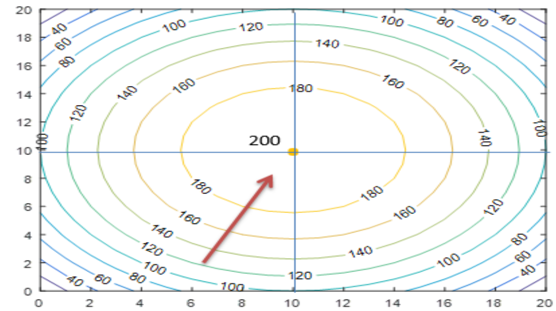
7.3.1) Statement of the problem with 2 variables and one single constraint

Unconstraint	Constraint
$\max_{x,y} (min)\, f(x, y; a)$	$\max_{x,y} (min)\, f(x, y; a)$ s.t. $g(x, y; a) = c$
Example: $\max_{x,y}\ a_0 - (x - a_1)^2 - (y - a_2)^2$	**Example:** $\max_{x,y}\ a_0 - (x - a_1)^2 - (y - a_2)^2$ $a_3x + a_4y = c$
Example: $\max_{x,y}\ 200 - (x - 10)^2 - (y - 10)^2$	**Example:** $\max_{x,y}\ 200 - (x - 10)^2 - (y - 10)^2$ $x + y = 10$
Note: *"a" and "c" are parameters*	

3-D curve of the objective function

2D representation of the objective function



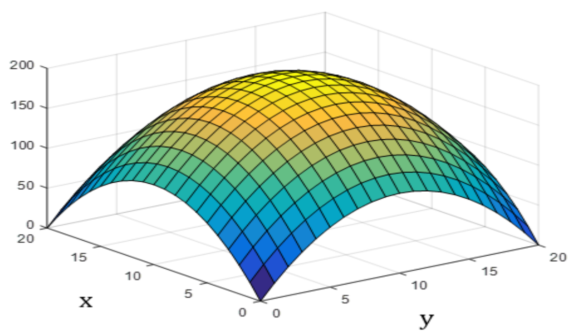


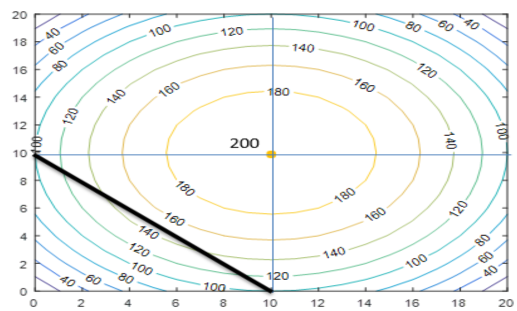
X= 10 and y = 10 is the optimum point under constraint-free optimization problem.

Introducing a constraint set would affect the possible choice set, and hence optimized value

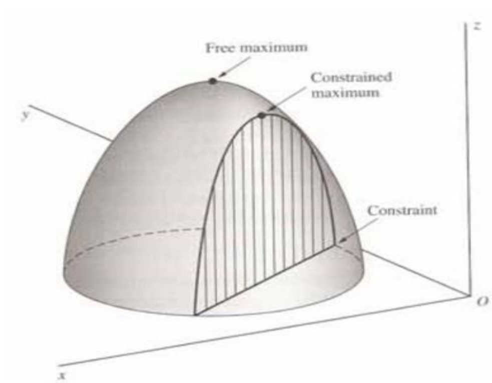
3-D curve of the objective function

2D representation of the objective function





Can we still choose x = 10 and y = 10 when the constraint is imposed?



- Lower the optimum value of the objective function.

- Our goal is to come up with solution methods that help finding the constrained optimum point.

7.3.1a) Substitution method

Example:

$$\max_{x,y}\ x * y$$

subject to $x + y = 7$

Limitations:

- Functional form of the constraint function is too complicate.
  - Ex. $\ln\left(\frac{x}{y}\right) + x^2 - \frac{y^2}{x^2+1} = \frac{3}{xy} + 2$
- More than single constraint set.
  - In general case, there can be more than one constraint.
  - Doable, but difficult/impractical to reduce the problem into the unconstrained optimization problem.

7.3.1b) LaGrange method

The method modifies our oringial problem into a so called "*<u>Psuedo</u>- unconstrained optimization*", where one instead maximizes or minimize an artificially constructed function so called the *LaGrange function*.

The *LaGrange function* takes the following form:

$$L(x, y, \lambda; a, c) = f(x, y; a) + \lambda[c - g(x, y; a)]$$

λ is the LaGrange multiplier, an auxiliary variable introduced for computational purpose, to be explained later.

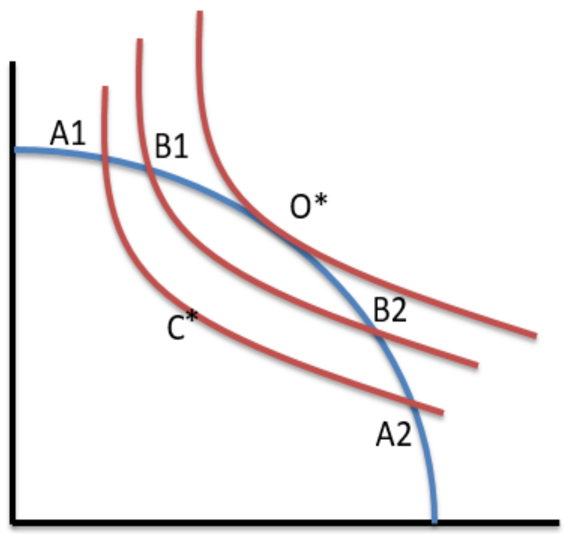
- Under some regularity conditions, the solution to the constrained optimization is the *stationary point* for the LaGrange function.
  - A stationary point of a function is the point where "first- order derivatives" are equal to zero.
- For the LaGrange function, they are (x^*, y^*, λ^*) such that

$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial \lambda} = 0$$

From the first-order condition, the basic idea of solution concept can be graphically illustrated below. Graphically, optimal solution is the one in which the "*objective function*" is tangent to the "*constraint equation*". That is,

(i) Slope of the objective function = Slope of the constraint equation
(ii) Objective function is the highest possible among the choice set imposed by the constraint.

In the figure, point-O* is the optimized solution to our problem.



- Keep moving the objective function until the function is tangent to the constraint set.

Example: Solve for the problem $\max_{x,y} x \quad y \quad$ *s.t.* $a_0x + a_1y = c$

The Interpretation of LaGrange multiplier

Definition: *Optimal value function*

$\mathrm{f}^*(\boldsymbol{a}, \mathbf{c}) = \mathrm{f}(x^*(a, \boldsymbol{c}), y^*(a, c))$ as the *optimal value* of the object function.

Envelop theorem:

Given the optimal value function,

$$\frac{d\mathrm{f}^*(a,\boldsymbol{c})}{dc} = \frac{dL(x,y;\boldsymbol{a},\boldsymbol{c})}{dc} \text{ and } \frac{d\mathrm{f}^*(a,\boldsymbol{c})}{da_i} = \frac{dL(x,y;\boldsymbol{a},\boldsymbol{c})}{da_i}$$

Proof:

Lemma:

Following the theorem, one knows that $\boldsymbol{\lambda}^*(\boldsymbol{a},\boldsymbol{c}) = \frac{\boldsymbol{df}^*(\boldsymbol{a},\boldsymbol{c})}{\boldsymbol{dc}}$

By approximation, we know that:

$$\mathrm{f}^*(\boldsymbol{a},\boldsymbol{c}+dc) - f^*(\boldsymbol{a},\boldsymbol{c}) \approx \lambda^*(\boldsymbol{a},\boldsymbol{c})dc$$

If we relax the constraint set/value (c) imposed on our optimization by a small amount of dc units, on its *first approximation*, the optimized value of the objective function would be changed by $\lambda^*(\boldsymbol{a},\boldsymbol{c})dc$.

Step 2: The second-order condition: confirming the result

- Similar to the unconstrained optimization problem, we need to confirm our result.
- If the objective function is <u>concave</u> at the stationary point (along the constraint set), solution to the first-order condition is then confirmed to be a constrained maximizer.
- To do this, we check for the property of the second-order derivative matrix of the LaGrange function.
- By checking the property of the so called "*Bordered*" Hessian matrix.

$$\bar{H} = \begin{bmatrix} L_{\lambda\lambda} & L_{\lambda x} & L_{\lambda y} \\ L_{x\lambda} & L_{xx} & L_{xy} \\ L_{y\lambda} & L_{yx} & L_{yy} \end{bmatrix}$$

Example: Check the concavity and convexity of the example above.

Checking conditions:

- For maximum, determinant of the bordered Hessian must be <u>POSITIVE.</u>
- For minimum, determinant of the bordered Hessian must be <u>NEGATIVE.</u>

Wrap's up: 2-step cooked procedure.

- Step 0: Forming the LaGrange function and
- Step 1: Derive stationary points of the LaGrange function (Solving the FOCs of LaGrange function)
- Step 2: Checking concavity/convexity of function along the constraint set.
    - Convex: Determinant of Bordered Hessian is NEGATIVE. (Minimum)
    - Concave: Determinant of Bordered Hessian is POSITIVE. (Maximum)

7.3.2) N variables and Single constraint

This part extends the analysis to the case when there are N choice varaibles, but with a single constraint. To make the presentation consiste, we label the argument as the variable indexed by "i", i.e. $x = \{x_i\}_{i=1}^{N} = (x_1, x_2, \dots, x_N)$. The problem is to maximize (minimize) the objective function, subject to a constraint.

$$\max_{x}(min)\, f(x_1, x_2, \dots, x_N; a)$$

s.t.
$$g(x_1, x_2, \dots, x_N; a) = c$$

Solving the given problem is easy. Let's form the LaGrange function.

$$L(x_1, x_2, \dots, x_N, \lambda; a, c) = f(x_1, x_2, \dots, x_N; a) + \lambda[\boldsymbol{c} - \boldsymbol{g}(x_1, x_2, \dots, x_N; a)]$$

The solution to the constrained optimization is to maximize (minimize) the LaGrange function instead. Using the first-order conditions, we yield that

FOC. Finding the stationary solution to the FOC conditions.

$$\frac{\partial L}{\partial x_1} = \frac{\partial L}{\partial x_2} = \cdots = \frac{\partial L}{\partial x_N} = \frac{\partial L}{\partial \lambda} = 0$$

SOC. Checking the second-derivative matrix of the LaGrange function

$$\bar{H} = \begin{bmatrix} L_{\lambda\lambda} & \cdots & L_{\lambda x_N} \\ \vdots & \ddots & \vdots \\ L_{x_N\lambda} & \cdots & L_{x_N x_N} \end{bmatrix}_{N+1\times N+1}$$

Checking condition:

(i) Check the "*last*" N-1 submatrix of the bordered heassian

(ii) Conditions are as follow;

a. Maximum; alternate sign, beginning with $(-1)^2$

b. Minimum; same sign, beginning with (-1)

Example: #N = 3 variable; $\bar{H}$: 4 x 4 → #N-1 submatrices = 3 – 1 = 2;

Leading minor matrices: $\bar{H}_1, \bar{H}_2, \bar{H}_3,$ and $\bar{H}_4 = \bar{H}$

The *last N-1 leading minor* matrices: $\bar{H}_3,$ and $\bar{H}_4 = \bar{H}$

Begning one: Sign($\bar{H}_3$)

Maximum: positive, then alternate.

Minimum: negative, then the same.

7.3.4) Some economics examples

Example: Regulated v.s. Unregulated monopolist

Suppose that a monopolist profit function is given by

$$\pi(x,y) = 64x - 2x^2 + 4xy - 4y^2 + 32y - 14$$

where "x" is the level of output sold to the first type of consumer and "y" is the level of output sold to the second type of consumer.

a) Find the optimal level of output that generate the highest profit

b) What happen if the government limits the total of quantity output to be equal to 79 units.

c) If the government allows for the production limit to be 80 units, what would happen to the optimized level of profit?

Example: Utility maximization problem (UMP)

Suppose that a household has the preference relationship defined by the utility function $U(x, y) = 2x^{0.6}y^{0.3}$. Suppose further that the price of goods x is P_x and the price of goods y is P_y. Given M as the budget of this household, find the optimal bundle of consumption for goods x and goods y.

Exercise: Redo the example with the new utility function

$$U(x,y) = (x-1)^{\frac{1}{2}}(y-4)^{\frac{1}{2}}.$$

The obtained solutions, $x^*(p_x, p_y, M)$ *and* $y^*(p_x, p_y, M)$, from the above utility maximization problem (UMP) yield us the so called "*Marshallian demand function*". The function is HM degree zero in prices and income.

Indirect utility function:

The optimized level of utility can be derived by plugging the optimal consumption bundle into the utility function. Economically, the optimized utility function, expressed in terms of prices and income, is called the ***indirect utility function***. Mathematically, the indirect utility function is the optimal value function attained from the optimization problem.

$$v(p_x, p_y, M) = U\left(\boldsymbol{x}^*(\boldsymbol{p_x}, \boldsymbol{p_y}, \boldsymbol{M}), \boldsymbol{y}^*(\boldsymbol{p_x}, \boldsymbol{p_y}, \boldsymbol{M})\right)$$

Properties:

(i) HM degree of zero in prices and income.

Proof:

(ii) $\boldsymbol{x}^* = -\frac{\frac{\partial v}{\partial p_x}}{\frac{\partial v}{\partial p_M}}$ and $\boldsymbol{y}^* = -\frac{\frac{\partial v}{\partial p_y}}{\frac{\partial v}{\partial p_M}}$ (Roy's identity)

Proof:

Compensated demand function (Hicksian demand curve)

If you have already studied EE311, you might have seen this concept before, i.e. the *Hicksian demand curve.*

An alternative form of the demand function can be derived from the *optimal bundle that minimizes the level of expenditure.* Hicks introduced this concept after Marhsall's introduction on demand function, and it has become known since then under the name of the *compensated demand function.*

Hicks's idea is that, instead of searching for the bundle that maximize the utility under budge, household is then assumed to *choose the bundle that minimizes the level of expenditure needed for acquiring a certain level of utility.* The problem under Hicks's supposition is then

$$\boldsymbol{mix\ \ p_x x + p_y y\ \ \ s.t.\, u(x, y) = \bar{u}}$$

Solution to the above Hicks's problem can be derived in a similar way to the problem stated out by Marshall. That, is we use the LaGrange method.

The resulting optimization problem yields us the solution where

$$\boldsymbol{x^h = x^h(p_x, p_y, \bar{u})\ \ \text{and}\ y^h = y^h(p_x, p_y, \bar{u})}$$

Given the Hicksian demand bundle, the optimized level of expenditure, commonly called the expenditure function, is given by,

$$\boldsymbol{e(p_x, p_y, \bar{u}) = p_x x^h(p_x, p_y, \bar{u}) + p_y y^h(p_x, p_y, \bar{u})}$$

Duality in Household decision problem

(i) $$x^h\left(p_x, p_y, v(p_x, p_y, M)\right) = x^*(p_x, p_y, M)$$

(ii) $$x^*\left(p_x, p_y, e(p_x, p_y, \bar{u})\right) = x^h(p_x, p_y, \bar{u})$$

(iii) Roy's Identity:

$$x^* = -\frac{\frac{\partial v}{\partial p_x}}{\frac{\partial v}{\partial p_M}} \text{ and } y^* = -\frac{\frac{\partial v}{\partial p_y}}{\frac{\partial v}{\partial p_M}}$$

(iv) Expenditure function:

$$x^h = \frac{\partial e(p_x, p_y, \bar{u})}{\partial p_x} \quad \text{and} \quad y^h = \frac{\partial e(p_x, p_y, \bar{u})}{\partial p_y}$$

Example: Cost function

Suppose that firm has the production technology given by $Q = \sqrt{K}+\sqrt{L}$. To acquire each unit of capital (K) and labor (L), firm must pay for the unit cost of r and w, respectively.

a) Derive the optimal combinations of factor inputs.

b) What is the long-run cost function?

c) If capital is fixed equal to K_o, rederive the (short-run) cost function?

Chapter 8 Integration and its application

8.1) Definition and Basic integration formula

- An **antiderivative** of a function f is a function F such that $F'(x) = f(x)$

  In differential notation, $dF = f(x)dx$

- Integration states that

  $\int f(x)dx = F(x) + C$ if only $F'(x) = f(x)$

- Basic Integration

  Properties:

1.	$\int k\,dx = kx + C$ $\quad k$ is a constant
2.	$\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \quad n \neq -1$
3.	$\int x^{-1}\,dx = \int \frac{1}{x}dx = \int \frac{dx}{x} = \ln x + C \quad$ for $x > 0$
4.	$\int e^x\,dx = e^x + C$
5.	$\int kf(x)\,dx = k\int f(x)\,dx$ $\quad k$ is a constant
6.	$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

Example: Find $\int \left(2\sqrt[5]{x^4} - 7x^3 + 10e^x - 1\right)dx$

Example: If y is a function of x such that dy/dx $= 8x - 4$ and $y(2) = 5$, find y.

Example: In the manufacture of a product, fixed costs per week are \$4000. (Fixed costs are costs, such as rent and insurance, that remain constant at all levels of production during a given time period.) If the marginal-cost function is

$$\frac{dc}{dq} = 0.000001\left(0.002q^2 - 25q\right) + 0.2$$

where c is the total cost (in dollars) of producing q pounds of product per week, find the cost of producing 10,000 lb in 1 week.

8.2) Definite integral

- If f is continuous on the interval $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

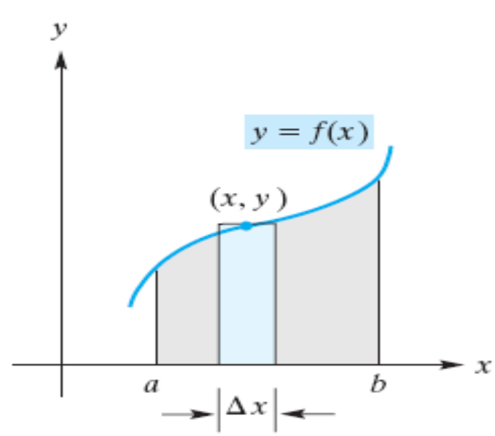
$$\int_a^b f(x)dx = F(b) - F(a)$$

- Property of definite integral

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,; \quad c \in [a, b]$$

- Definite integral as the area under curve
  - Suppose we are interested in finding the area under f(x) bounded by (over the interval) [a,b].
  - Geometrically.
    - The width of the vertical element is Δx. The height is the y-value of the curve.
  - The area is defined as

$$\sum f(x)\Delta x \to \int_a^b f(x)dx = area$$



Example: Find the area of the region bounded by the curve $y = 6 - x - x^2$ for $x \in [-3,2]$

Example: Find the area of the region under the curve $y = x^3$ for $x \in [-2,2]$

8.3) Applications of integration in economics.

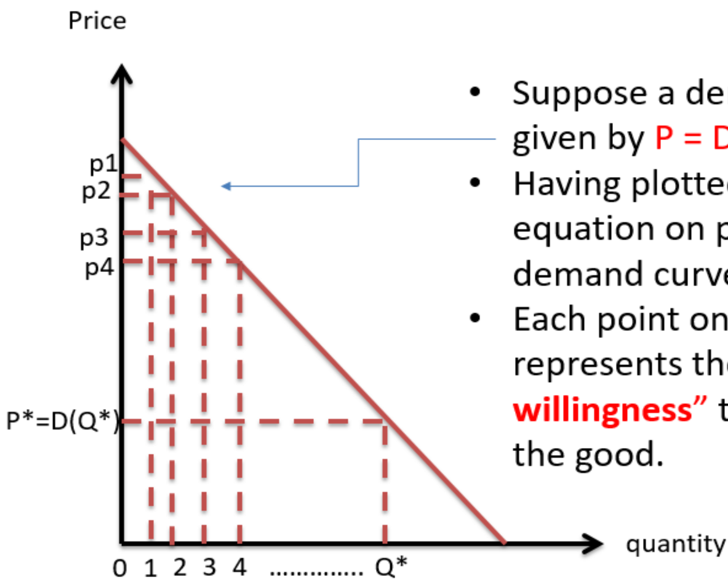
Recoverability: getting back to the level function from the marginal function

- $MU(x) \rightarrow U(x)$
- $MC(Q) \rightarrow C(Q)$
- Invement as a change in capital stock!: $I(t) = \frac{dK(t)}{dt} \rightarrow K(t)$

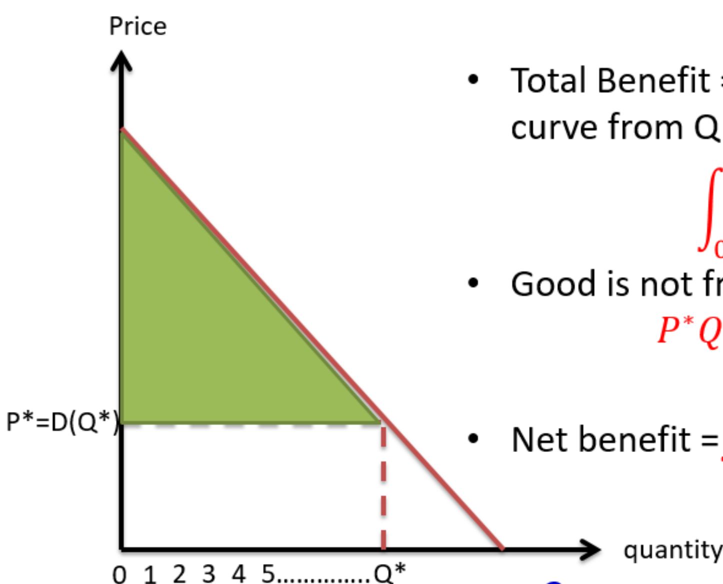
Welfare analysis:

- We normally want to measure the desirability of market outcome or market allocation.
- There are several welfare measurements
- Consumers' and producers' surplus (CS/PS) are the most popularized concept that most economists use.
  - Given information about demand and supply curve, we can measure CS and PS by applying the (definite) integration as the tool for measurement.
- What are they?

Consumer surplus



- Suppose a demand function is given by P = D(Q).
- Having plotted the demand equation on p-q axis, we get a demand curve.
- Each point on the demand curve represents the **"maximum willingness"** to pay for each unit of the good.



- Total Benefit = Area under the demand curve from Q = 0 upto Q = Q*

$$\int_0^{Q^*} D(Q)dQ$$

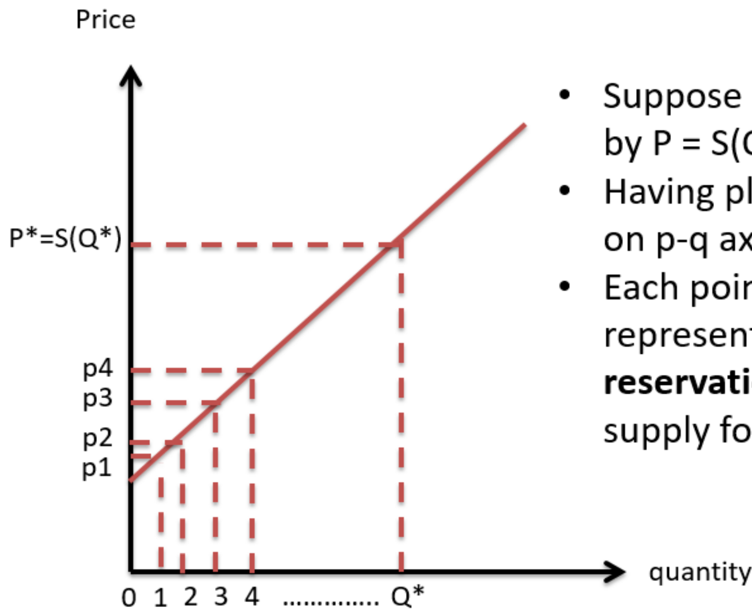
- Good is not free!

$$P^*Q^* = D(Q^*)Q^*$$

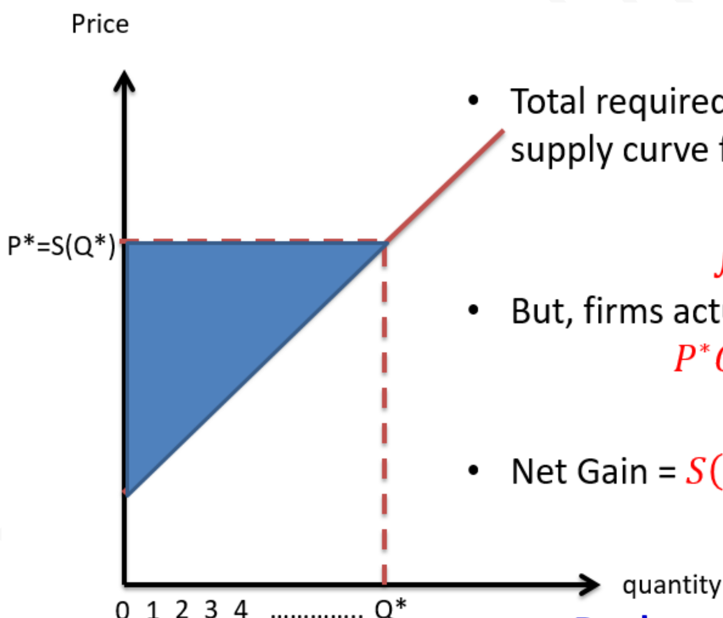
- Net benefit = $\int_0^{Q^*} D(Q)dQ - D(Q^*)Q^*$

Consumers' surplus =
Area under demand, but above price

Producer surplus



- Suppose a supply function is given by P = S(Q).
- Having plotted the supply equation on p-q axis, we get a supply curve.
- Each point on the supply curve represents the "**minimum reservation price**" required to supply for each unit of the good.



- Total required revenue = Area under the supply curve from Q = 0 upto Q = Q*

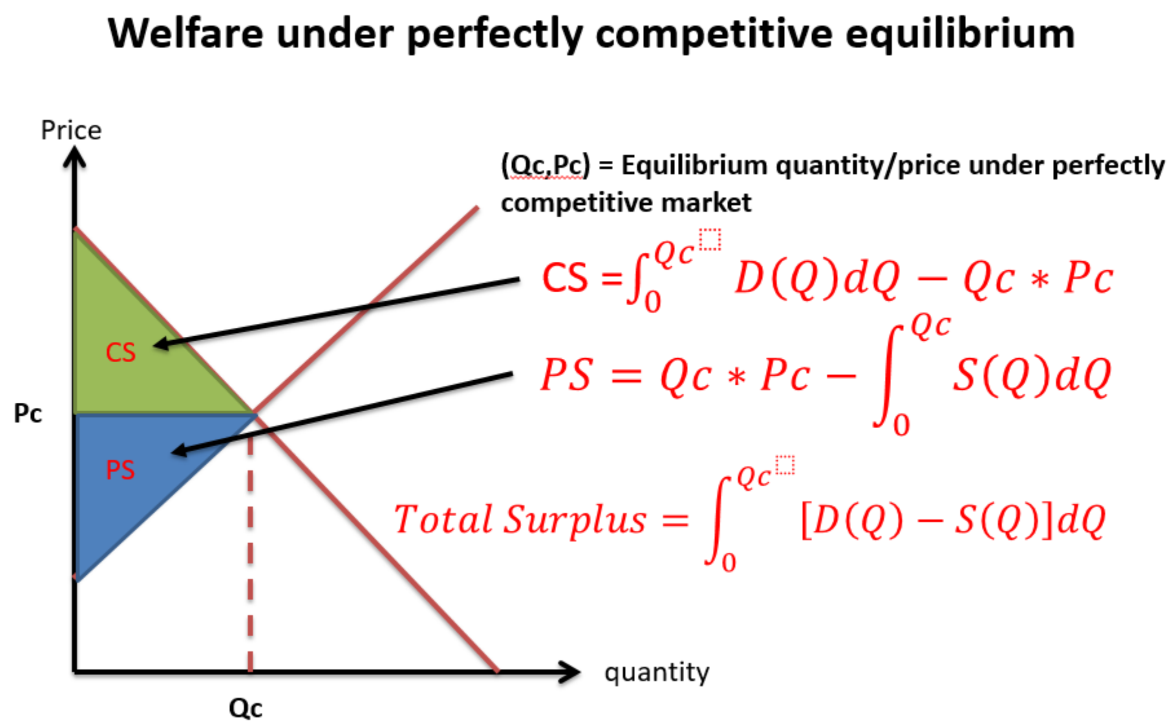
$$\int_0^{Q^*} S(Q)dQ$$

- But, firms actually get

$$P^*Q^* = S(Q^*)Q^*$$

- Net Gain = $S(Q^*)Q^* \; - \int_0^{Q^*} S(Q)dQ$

Producers' surplus = area above supply, but under price


Welfare under perfectly competitive equilibrium
Price
(Qc,Pc) = Equilibrium quantity/price under perfectly competitive market
$CS = \int_0^{Qc} D(Q)dQ - Qc * Pc$
CS
$PS = Qc * Pc - \int_0^{Qc} S(Q)dQ$
Pc
PS
$Total\ Surplus = \int_0^{Qc} [D(Q) - S(Q)]dQ$
quantity
Qc

Example: Suppose the market demand equation is $Q = \frac{1}{2}(25 - P)$ and the market supply equation is $Q = \frac{1}{2}(P - 1)$. Evaluate the welfare at the market equilibrium price and quantity.

First-best equilibrium:

- Economist typically refers to the "perfectly competitive" equilibrium/allocation as the benchmark for the first-best equilibrium.
  - Proof should be given in Micro311, for the part that discusses about General Equilibrium.
  - Rigorous proof can be found in Debreu (1959) "*Theory of value*".
- In what sense?
  - In the sense that the allocation generates the biggest total surplus of the two (consumers/producers) combined.
- Any market inventions on the perfectly competitive market would do more harm than good.
  - Any interventions, such as taxes/subsidies/regulations, would do harm, rather than doing good, e.g. yielding us lowered welfare
- Any markets with the structure deviating from the assumptions imposed under perfectly competitive market would yield lower welfare.

Example: Suppose that the market demand is given by $P = 30 - Q$ and the cost function is given by $C(Q) = \frac{1}{2}Q^2$. Calculate the Deadweight loss that arises under monopoly market.

8.4) Basic differential equation (optional)

Suppose that $y = f(t)$, but don't know the function form of "f".

However, we know that **"y"** satisfies the equation given by a function

$$G(y, y', y'', y''', \ldots, y^n, t) = 0$$

where $y^n = \frac{d^n y}{dt^n}$ → n-th derivative.

The problem that we attempt to uncover the unknown form of "f" by using the information obtained in " G" is mathematically called the **differential equation problem!**

For example:

$G(y', y'', y''', \ldots, y^n) = y' + 2y - 3 + 4t$

- first-order differential equation (linear)

$G(y', y'', y''', \ldots, y^n) = y' + 2y - 3$

- first-order differential equation (linear/autonomous case)

$G(y', y'', y''', \ldots, y^n) = y' + 2y$

- first-order differential equation (linear/autonomous case/ homogenous case)

$G(y', y'', y''', \ldots, y^n) = y'' + 3y' - \frac{1}{5}y$

- second-order differential equation (linear/autonomous case/homogenous case)

$G(y', y'', y''', \ldots, y^n) = (y'')(y') + 3y' - \frac{1}{5}y$

- second-order differential equation (non-linear system; second-degree)

General form: **The n-order linear differential equation**

$$a_n y^n + a_{n-1} y^{n-1} + \cdots + a_2 y'' + a_1 y' + a_0 y + k + bt = 0$$

- If k = 0 → Homogenous case
- If b = 0 → Autonomous case
- n = 1 → firs-order equation

In this lecture, we provide some basic tools that can be used to solve for the solution of **first-order homogenous linear differential equation**.

$$a_1 y' = a_0 y + k + bt$$

Case 1: Solution to Homogenous and autonomous equation ($k = 0, b = 0$)

Case 2: Solution to non-homogenous autonomous equation ($b = 0$)

The solution takes the form: $y = y_h + y_s$

where

(i) $y_h =$ Solution to the homogenous equation (when k = 0)

(ii) $y_s =$ Solution to the steady-state equation (what is it?)

Stability of the solution:

The solution is said to be stable if $\lim_{t\to\infty} y(t) \to y_s\ (steady\ solution)$

What condition do we need?