Solution Part III: Exercise for Assignment 5

1. Let $X = \{2, 3, 4\}$ and $Y = \{1, 5\}$. Define relations
   $f : X \to Y$ by $f = \{(x, y) \in X \times Y | x > y\}$,
   $g : X \to Y$ by $g = \{(x, y) \in X \times Y | x = 2y\}$, and
   $h : X \to Y$ by $h = \{(x, y) \in X \times Y | y = x^2\}$.

   (a) List all the elements of the Cartesian product $X \times Y$.

   (b) List all the elements of the relations f, g, and h.

   (c) Determine which of the relations f, g, h is a function from X to Y. Explain your answer.

   (d) What is the inverse image of 1 under f?

   (e) What is the inverse image of 5 under f?

   Solution:

   (a) List all the elements of the Cartesian product $X \times Y$.

   $$X \times Y = \{(2, 1), (2, 5), (3, 1), (3, 5), (4, 1), (4, 5)\}$$

   (b) List all the elements of the relations f, g, and h.

   $$f = \{(2, 1), (3, 1), (4, 1)\}.$$
   $$g = \{(2, 1)\}$$
   $$h = \{\} = \emptyset$$

   (c) Determine which of the relations f, g, h is a function. Explain your answer.

   -The relation f is a function because all the elements in the domain X get mapped to an element in Y and each of them gets mapped only once (even though the element 1 in Y gets mapped more than once).

   -The relation g is not a function from X to Y, since there are some element in X doesn't get mapped to Y(i.e. $x = 3, 4$).

   -The relation h is not a function from X to Y, since none of the elements in X gets mapped (i.e. $x = 2, 3, 4$) to Y.

   (d) What is the inverse image of 1 under f?

   $$\{x \in X | (x, 1) \in f\} = \{2, 3, 4\}$$

   (e) What is the inverse image of 5 under f?

   $$\{x \in X | (x, 5) \in f\} = \emptyset$$

2. Let $A = \{1, 2, 3\}$ and let $\mathcal{P}(A)$ be the set of all subsets of the set A. Define a relation r and s as

$$r = \{(x, y) \in A \times \mathcal{P}(A) \mid \ x = \text{ the number of elements in } y\},$$

$$s = \{(u, v) \in \mathcal{P}(A) \times A \mid (v, u) \in r\}.$$

   (a) Draw arrow diagrams of r and s.

   (b) Is r a function? If so, is it onto and/or one-to-one? Justify your answer.

   (c) Is s a function? If so, is it onto and/or one-to-one? Justify your answer.

Solution:

   (a) Draw an arrow diagrams of r and s.

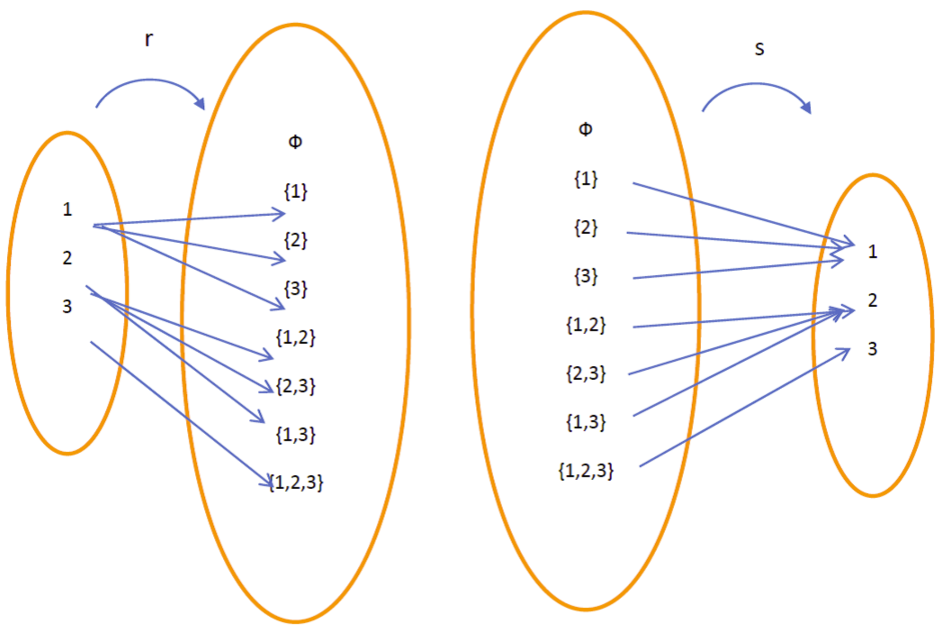

Figure 1: Problem 2(a)

   (b) Is r a function? If so, is it onto and/or one-to-one? Justify your answer.
   No, r is a not a function because the elements 1 in A gets map $\{1\}, \{2\}, \{3\} \in \mathcal{P}(A)$(i.e. it gets mapped more than once and similarly for $2 \in A$).

   (c) Is s a function? If so, is it onto and/or one-to-one? Justify your answer.
   No, s is a not a function because the elements $\emptyset \in \mathcal{P}(A)$ does not get mapped to any element in A.

3. Define $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows:

$$H(x, y) = (y, x - 2) \text{ for all } (x, y) \in \mathbb{R} \times \mathbb{R}.$$

   (a) Is H one-to-one? Prove or give a counterexample.

(b) Is H onto? Prove or give a counterexample.

(c) Is H bijective? If so, find H^{-1}, the inverse function of H.

Solution:

(a) Is H one-to-one? Prove or give a counterexample.
Solution: Yes, H is one-to-one. Let $(x_1, y_1), (x_2, y_2)$ be some elements in the domain $\mathbb{R}\times\mathbb{R}$.
We want to show that if $H(x_1, y_1) = H(x_2, y_2)$, then $(x_1, y_1) = (x_2, y_2)$.
Suppose $H(x_1, y_1) = H(x_2, y_2)$. Then

$$(y_1, x_1 - 2) = (y_2, x_2 - 2)$$

or equivalently, $y_1 = y_2$ and $x_1 - 2 = x_2 - 2$. I.e.,

$$x_1 - 2 = x_2 - 2 \qquad \Rightarrow \qquad x_1 = x_2$$

That is, $H(x_1, y_1) = H(x_2, y_2)$ implies $(x_1, y_1) = (x_2, y_2)$ and therefore, H is one-to-one.

(b) Is H onto? Prove or give a counterexample.
Solution: Yes, H is not onto. Notice that if we pick (u, v) from the co-domain $\mathbb{R} \times \mathbb{R}$ and suppose that there is (x, y) in the domain such that $H(x, y) = (u, v)$, then

$$(u, v) = H(x, y) = (y, x - 2\ )$$

or we must have

$$u = y \quad \Rightarrow \quad y = u \in \mathbb{R}$$

and

$$v = x - 2 \quad \Rightarrow \quad x = v + 2 \in \mathbb{R}.$$

That is, for a given (u, v) from the co-domain $\mathbb{R} \times \mathbb{R}$ we can use $(x, y) = (v + 2, u)$ so that

$$H(x, y) = H\,(v + 2, u) = (u, (v + 2) - 2\ ) = (u, v).$$

Therefore, H is onto.

(c) Is H bijective? If so, find H^{-1}, the inverse function of H.
Solution: Yes.
From (a) and (b), since H is both one-to-one and onto, then H bijective.
To find the inverse function, H^{-1}, of H, we recall from the definition

$$H^{-1}(u, v) = (x, y) \qquad \Leftrightarrow \qquad H(x, y) = (u, y).$$

Since we have from (b) that

$$H\,(v + 2, u) = (u, v),$$

and hence the inverse function $H^{-1} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is given by

$$H^{-1}(u, v) = (v + 2, u)\,.$$

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4. Define functions f and g as follows:
$f = \{(1,10),(3,30),(5,50)\}$ and
$g = \{(10,k),(20,\ell),(30,m),(40,n),(50,t)\}$.

   (a) Determine the domain and range for each of functions f and g.
   (b) Find $g \circ f$ and $f \circ g$ (if possible) and the corresponding domain and range for each of them.
   (c) Find $g \circ g^{-1}$ and $g^{-1} \circ g$ (if possible) and their corresponding domains and ranges.

Solution:

(a) Determine the domain and range for each of functions f and g.

$D_f = \{1,3,5\}, \quad R_f = \{10,30,50\},$
$D_g = \{10,20,30,40,50\}, \quad R_g = \{k,\ell,m,n,t\}$

(b) Find $g \circ f$ and $f \circ g$ (if possible) and the corresponding domain and range for each of them.
$g \circ f$: Since $R_f \cap D_g = \{10,30,50\} \neq \emptyset$, we can construct $g \circ f$:
$(g \circ f)(1) = g(f(1)) = g(10) = k$
$(g \circ f)(3) = g(f(3)) = g(30) = m$
$(g \circ f)(5) = g(f(5)) = g(50) = t.$

That is, $g \circ f = \{(10,k),(30,k),(50,t)\}$.
The domain is $\{10,30,50\}$ and the range is $\{k,m,t\}$.
$f \circ g$: Since $R_g \cap D_f = \{k,\ell,m,n,t\} cap \{1,3,5\} = \emptyset$, we cannot construct $f \circ g$.

(c) Find $g \circ g^{-1}$ and $g^{-1} \circ g$ (if possible) and their corresponding domains and ranges.
First notice that g^{-1} is a function since g is bijective.

$$g^{-1} = \{(k,10),(\ell,20),(m,30),(n,40),(t,50)\}.$$

That is , $D_{g^{-1}} = \{k,\ell,m,n,t\}$ and $R_{g^{-1}} = \{10,20,30,40,50\}$.

$g \circ g^{-1}$: Since $R_{g^{-1}} \cap D_g = \{10,20,30,40,50\} \neq \emptyset$, we can construct $g \circ g^{-1}$:
$(g \circ g^{-1})(k) = g(g^{-1}(k)) = g(10) = k$
$(g \circ g^{-1})(\ell) = g(g^{-1}(\ell)) = g(20) = \ell$
$(g \circ g^{-1})(m) = g(g^{-1}(m)) = g(30) = m$
$(g \circ g^{-1})(n) = g(g^{-1}(n)) = g(40) = n$
$(g \circ g^{-1})(t) = g(g^{-1}(t)) = g(50) = t.$
That is,

$$g \circ g^{-1} = \{(k,k),(\ell,\ell),(m,m),(n,n),(t,t)\}.$$

That is, $D_{g \circ g^{-1}} = R_{g \circ g^{-1}} = \{k,\ell,m,n,t\}$.
$g^{-1} \circ g$: Since $D_{g^{-1}} \cap R_g = \{k,\ell,m,n,t\} \neq \emptyset$, we can construct $g \circ g^{-1}$:
$(g^{-1} \circ g)(10) = g^{-1}(g(10)) = g^{-1}(k) = 10$
$(g^{-1} \circ g)(20) = g^{-1}(g(20)) = g^{-1}(\ell) = 20$
$(g^{-1} \circ g)(30) = g^{-1}(g(30)) = g^{-1}(m) = 30$
$(g^{-1} \circ g)(40) = g^{-1}(g(40)) = g^{-1}(n) = 40$

$(g^{-1} \circ g)(50) = g^{-1}(g(50)) = g^{-1}(t) = 50.$
That is,

$$g^{-1} \circ g = \{(10,10), (20,20), (30,30), (40,40), (50,50)\}.$$

That is, $D_{g^{-1}\circ g} = R_{g^{-1}\circ g} = \{10, 20, 30, 40, 50\}$.

5. Define

$$g(x) = \frac{1}{\sqrt{x+1}} + 1 \qquad \text{and} \qquad F(x) = \begin{cases} x^2 - 1, & x \in [-3, 1) \\ 2 - x, & x \in [1, 4]. \end{cases}$$

(a) Find the domain and range for each of the functions g and F.

(b) Construct the composite functions $F \circ g$, and $g \circ F$ (if possible). Determine the domains for these composite functions.

(c) Is the function F bijective? If so, find the **inverse function** of F. Justify your answer.

Solution:

(a) Find the domain, co-domain, and range for each of the functions g and F.

To find the domain of g, notice that we must have $x+1 > 0$ or $x > -1$. That is, $x \in (-1, \infty)$.
To find the range of g,

$$x + 1 > 0 \quad \Rightarrow \quad \sqrt{x+1} > 0 \quad \Rightarrow \quad \frac{1}{\sqrt{x+1}} > 0 \quad \Rightarrow \quad \frac{1}{\sqrt{x+1}} + 1 > 1 \quad \Rightarrow \quad g(x) > 1$$

That is,

$$D_g = (-1, \infty), \qquad \text{and} \qquad R_g = (1, \infty).$$

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To find the domain of F, notice that $x^2 - 1$ is well-defined for $x \in [-3, 1)$ and $2 - x$ is well-defined for $x \in [1, 4]$.
Hence the domain $D_F = [-3, 4]$.
To find the range of F, we consider 2 cases.

(i) Suppose $x \in [-3, 1)$.
If $x \in [-3, -1]$, $-3 \leq x \leq -1 \quad \Rightarrow \quad 1 \leq x^2 \leq 9 \;\Rightarrow\; 0 \leq x^2 - 1 \leq 8 \;\Rightarrow\; F(x) \in [0, 8]$.

If $x \in [-1, 1)$, $-1 \leq x < 1 \quad \Rightarrow \quad 0 \leq x^2 \leq 1 \;\Rightarrow\; -1 \leq x^2 - 1 \leq 0 \;\Rightarrow\; F(x) \in [-1, 0]$.
That is, $F(x) \in [-1, 0] \cup [0, 8] = [-1, 8]$ when $x \in [-3, 1)$.

(ii) Suppose $x \in [1, 4]$. Then

$$1 \leq x \leq 4 \quad \Rightarrow \quad -4 \leq -x \leq -1 \quad \Rightarrow \quad -2 \leq 2 - x \leq 1 \quad \Rightarrow \quad F(x) \in [-2, 1].$$

Hence $F(x) \in [-1, 8] \cup [-2, 1] = [-2, 8]$ and the range of F is $R_F = [-2, 8]$.
That is,

$$D_g = (-1, \infty), \qquad \text{and} \qquad R_g = (1, \infty).$$
$$D_F = [-3, 4], \qquad \text{and} \qquad R_F = [-2, 8].$$

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(b) Construct the composite functions $F \circ g$, and $g \circ F$ (if possible). Determine the domains for these composite functions.

<u>$F \circ g$</u>: Notice that $R_g \cap D_F = (1, \infty) \cap [-3, 4] \neq \emptyset$. So, we can construct $F \circ g$:

$$(F \circ g)(x) = F(g(x)) = F\left(\frac{1}{\sqrt{x+1}} + 1\right) = 2 - \left(\frac{1}{\sqrt{x+1}} + 1\right) = 1 - \frac{1}{\sqrt{x+1}}.$$

Above, we have used the fact that $\frac{1}{\sqrt{x+1}} + 1 \in R_g = (1, \infty)$ and $\frac{1}{\sqrt{x+1}} + 1 \geq 1$ which implies that the formula in the second condition of F has to be used. To find $D_{F \circ g}$, first note that $D_{F \circ g} \subset D_g = (-1, \infty)$ and we must also have that $x \in D_{F \circ g}$ must satisfy $g(x) \in D_F \cap R_g$

$$\Rightarrow \quad \frac{1}{\sqrt{x+1}} + 1 \in [-3, 4] \cap (1, \infty) \quad \Rightarrow \quad 1 < \frac{1}{\sqrt{x+1}} + 1 \leq 4 \quad \Rightarrow \quad 0 < \frac{1}{\sqrt{x+1}} \leq 3 \quad \Rightarrow$$

$$\sqrt{x+1} \geq \frac{1}{3} \quad \Rightarrow \quad x + 1 \geq \frac{1}{9} \quad \Rightarrow \quad x \geq -\frac{8}{9}$$

and the domain of $D_{F \circ g} = [-\frac{8}{9}, \infty) \cap (-1, \infty) = [-\frac{8}{9}, \infty)$.

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<u>$g \circ F$</u>: Notice that $R_F \cap D_g = [-2, 8] \cap (-1, \infty) \neq \emptyset$. So, we can construct $g \circ F$: for $x \in D_F = [-3, 4]$,

$$(g \circ F)(x) = g(F(x)).$$

(i) For $x \in [-3, 1)$,

$$(g \circ F)(x) = g(F(x)) = g(x^2 - 1) = \frac{1}{\sqrt{(x^2 - 1) + 1}} + 1 = \frac{1}{\sqrt{x^2}} + 1 = \frac{1}{|x|} + 1.$$

The value of $x \in [-3, 1)$ must also satisfy $F(x) \in (-1, \infty)$ or $x^2 - 1 > -1 \Rightarrow x^2 > 0$. Hence, $x \in [-3, 1) - \{0\} = [-3, 0) \cup (0, 1)$.

(ii) For $x \in [1, 4]$,

$$(g \circ F)(x) = g(F(x)) = g(2 - x) = \frac{1}{\sqrt{(2 - x) + 1}} + 1 = \frac{1}{\sqrt{3 - x}} + 1.$$

The value of $x \in [1, 4]$ must also satisfy $F(x) \in (-1, \infty)$ or $2 - x > -1 \Rightarrow x < 3$. Hence, $x \in [1, 4] \cap (-\infty, 3) = [1, 3)$.

That is, from (i) and (ii)

$$(g \circ F)(x) = \begin{cases} \frac{1}{|x|} + 1, & x \in [-3, 0) \cup (0, 1) \\ \frac{1}{\sqrt{3-x}} + 1, & x \in [1, 3). \end{cases}$$

and the domain is $D_{g \circ F} = [-3, 0) \cup (0, 1) \cup [1, 3) = [-3, 0) \cup (0, 3)$. ■

(c) Is the function F bijective? If so, find the **inverse function** of F. Justify your answer.

No. Since F is not injective. E.g. when $x = \pm\frac{1}{\sqrt{2}} \in [-3, 1) \in D_F$,

$F(-\frac{1}{\sqrt{2}}) = F(\frac{1}{\sqrt{2}}) = \frac{1}{2} - 1 = -\frac{1}{2}$ but $-\frac{1}{\sqrt{2}} \neq \frac{1}{\sqrt{2}}$.

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