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ASSIGNMENTKEY #2.

Name ________ Surname ________ Student ID. ________

DUE DATE : Thursday 18^{th}, February 2016.

Assignment 2: (50 marks)

I pledge to the Honor Code and to obey all rules for taking and performing homework assignments as specified by the course instructor.

Student Signature: ________

1. (10 marks) Let X be a random variable with probability density function

$$f(x) = \begin{cases} c(1-x^2), & \text{if } -1 < x < 1. \\ 0, & \text{otherwise.} \end{cases} \qquad \text{(Eq.1)}$$

(a) What is the value of c?

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$$

$$\int_{-\infty}^{-1} \cancel{f(x)}^{0}\,dx + \int_{-1}^{1} f(x)\,dx + \int_{1}^{\infty} \cancel{f(x)}^{0}\,dx = 1$$

$$\int_{-1}^{1} c(1-x^2)\,dx = 1 \qquad cx - \frac{cx^3}{3}\Big|_{-1}^{1} = 1$$

$$\int_{-1}^{1} (c - cx^2)\,dx = 1 \qquad \left(c - \frac{c}{3}\right) - \left(-c + \frac{c}{3}\right) = 1$$

$$2c - \frac{2c}{3} = 1 \;\therefore\; \frac{6c - 2c}{3} = 1 \rightarrow c = \frac{3}{4} \;\#$$

(b) P (-0.5< X<0.5)

$$P(-0.5 < X < 0.5) = \int_{-0.5}^{0.5} \frac{3}{4}(1-x^2)\,dx = \frac{3}{4}\left(x - \frac{x^3}{3}\right)\Big|_{-0.5}^{0.5}$$

$$= \frac{3}{4}\int_{-0.5}^{0.5}(1-x^2)\,dx = \frac{3}{4}\left[\left(0.5 - \frac{0.5^3}{3}\right) - \left(-0.5 - \frac{(-0.5)^3}{3}\right)\right]$$

2. (10 marks)Determine whether the following models are linear in the parameters, or the variables, or both. Which of these models are linear regression models?

2.1 Reciprocal

- linear in parameter
- linear regression model

$$Y_i = \beta_1 + \beta_2\left(\frac{1}{X_i}\right) + u_i$$

2.2 Semilogarithmic

- linear in parameter
- linear regression model

$$Y_i = \beta_1 + \beta_2 \ln X_i + u_i$$

2.3 Inverse semilogarithmic

- linear in parameter
- linear regression model

$$\ln Y_i = \beta_1 + \beta_2 X_i + u_i$$

$$Y_i = e^{\beta_1 + \beta_2 X_i + u_i}$$

2.4 Logarithmic or double logarithmic

- linear in parameter
- linear regression model

$$\ln Y_i = \ln \beta_1 + \beta_2 \ln X_i + u_i$$

$$Y_i = (\beta_1 X_i^{\beta_2} \cdot e^{u_i})$$

2.5 Logarithmic reciprocal

- linear in parameter
- linear regression model

$$\ln Y_i = \beta_1 - \beta_2\left(\frac{1}{X_i}\right) + u_i$$

$$Y = e^{\beta_1 - \beta_2 \left(\frac{1}{X_i}\right) + u_i}$$

3. (10 marks)Consider the following non-stochastic models (i.e., model without the stochastic error term). Are they linear regression models? If not, is it possible, by suitable algebraic manipulations, to convert them into linear models?

3.1

$$Y_i = \frac{1}{\beta_1 + \beta_2 X_i}$$

$$\frac{1}{Y_i} = \beta_1 + \beta_2 X_i$$

3.2

$$Y_i = \frac{X_i}{\beta_1 + \beta_2 X_i}$$

$$\frac{Y_i}{X_i} = \frac{1}{\beta_1 + \beta_2 X_i}$$

$$\frac{X_i}{Y_i} = \beta_1 + \beta_2 X_i$$

3.3

$$Y_i = \frac{1}{1 + exp(-\beta_1 - \beta_2 X_i)}$$

$$Y_i = \frac{1}{1 + e^{(-\beta_1 - \beta_2 X_i)}}$$

$$\frac{1}{Y_i} = 1 + e^{(-\beta_1 - \beta_2 X_i)}$$

$$\frac{1}{Y_i} - 1 = e^{(-\beta_1 - \beta_2 X_i)}$$

$$\frac{1 - Y_i}{Y_i} = e^{(-\beta_1 - \beta_2 X_i)}$$

$$\ln\left(\frac{1 - Y_i}{Y_i}\right) = -\beta_1 - \beta_2 X_i$$

$$-\ln\left(\frac{1 - Y_i}{Y_i}\right) = \beta_1 + \beta_2 X_i \ \#$$

$\therefore$ All can be transformed into LRM.

4. (10 marks)(Mid-term Exam 1/2015) Show that $\sum x_i^2 = \sum X_i^2 - n\bar{X}^2$ and that $\sum x_i y_i = \sum x_i Y_i$. In other words, show that:

$$\hat{\beta}_2 = \frac{\sum x_i y_i}{\sum x_i^2} = \frac{\sum x_i Y_i}{\sum X_i^2 - n\bar{X}^2}$$

$$\hat{\beta}_2 = \frac{\sum xy}{\sum x^2} = \frac{\sum(X-\bar{X})(Y-\bar{Y})}{\sum(X-\bar{X})^2}$$

$$= \frac{\sum([XY - Y\bar{X}] - X\bar{Y} + \bar{X}\bar{Y})}{\sum(X^2 - 2X\bar{X} + \bar{X}^2).}$$

$$= \frac{\sum(XY) - \bar{Y}\sum X + n\bar{X}\bar{Y}}{\sum(X^2) - 2\bar{X}\sum X + n\bar{X}^2} \qquad \bar{X} = \frac{\sum X}{n} \quad \therefore n\bar{X} = \sum X$$

$$= \frac{\sum(XY) - n\bar{Y}\bar{X} + n\bar{X}\bar{Y}}{\sum(X^2) - 2\bar{X}(n\bar{X}) + n\bar{X}}$$

$$\hat{\beta}_2 = \frac{\sum XY}{\sum X^2 - n\bar{X}^2}$$

Table 2. Raw Data Based on the Sample Data on Table 1

	(1) Y_i	(2) X_i	(3) Y_iX_i	(4) X_i^2	(5) $x_i = X_i - \bar{X}$	(6) $y_i = Y_i - \bar{Y}$	(7) x_i^2	(8) x_iy_i	(9) $\hat{Y}_i$	(10) $\hat{u}_i = Y_i - \hat{Y}_i$	(11) $\hat{Y}_i\hat{u}_i$
	2.8	63	176.4	3969	-14.625	-0.4125	213.8906	6.03	2.71	0.09	0.23
	3.4	72	244.8	5184	-5.625	0.1875	31.64063	-1.055	3.02	0.35	1.15
	3	78	234	6084	0.375	-0.2125	0.140625	-0.08	3.23	-0.23	-0.73
	3.5	81	283.5	6561	3.375	0.2875	11.39063	0.97	3.33	0.17	0.57
	3.6	87	313.2	7569	9.375	0.3875	87.89063	3.63	3.53	0.07	0.24
	3	75	225	5625	-2.625	-0.2125	6.890625	0.56	3.12	-0.12	-0.38
	2.7	75	202.5	5625	-2.625	-0.5125	6.890625	1.35	3.12	-0.42	-1.32
	3.7	90	333	8100	12.375	0.4875	153.1406	6.03	3.63	0.07	0.24
Sum	25.7	621	2012.4	48717	0	0	511.875	17.44	25.7	0	0
Mean	3.2125	77.625	251.55	6089.625	0	0	63.984375	2.18	3.2125	0	0

5.1 Now consider the two-variable model :

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

Based on Table 2, use the OLS to find the estimator of β_1 and β_2. Interpret the regression.

Estimator of β_2 is $\hat{\beta}_2$ then;

$$\hat{\beta}_2 = \frac{\sum(x-\bar{x})y}{\sum(x-\bar{x})^2} = \frac{\sum(x-\bar{x})y}{SST}$$

$$= \frac{(8)}{(7)} = \frac{17.44}{511.575}$$

$$\hat{\beta}_2 = 0.34071$$

Estimator of β_1 is $\hat{\beta}_1$ then;

$$\hat{\beta}_1 = \bar{y} - \hat{\beta}_2 \bar{x}$$

$$\hat{\beta}_1 = 3.2125 - (0.034071)(77.625)$$

$$= 0.567753$$

5.2 Show that $\sum \hat{u}_i = 0$

For OLS; $\min_{\beta} \sum \hat{u}_i^2 = \min_{\beta} \sum(y_i - \hat{y}_i)^2 = \min_{\beta} \sum(y_i - x^T\beta)^2$

FOC; $\nabla_{\beta} \sum \hat{u}_i^2 = \begin{bmatrix} -2\sum(y-\hat{\beta}_1-\hat{\beta}_2 x) \\ -2\sum(y-\hat{\beta}_1-\hat{\beta}_2 x)x \end{bmatrix} = \underline{0}$

For the first row; $-2\sum(y-\hat{\beta}_1-\hat{\beta}_2 x) = 0$

$\sum(y-\hat{\beta}_1-\hat{\beta}_2 x) = 0$

which $\sum\hat{u}_i = \sum(y-\hat{y}) = 0$

Hence $\sum\hat{u}_i = 0$.