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Question 1:

Given the equation for the production function

$$Q = f(K, L) = 18 * [0.2K^{-0.4} + 0.8L^{-0.4}]^{-2.5}$$

1.1 What type of constant return to scale does the production function exhibit?

1.2 Is the production function increasing with respect to K and L?

1.3 Use the implicit function rule to find the marginal rate of technical substitution (MRTS) of L for K.

1.1. $Q = f(K,L) = 18 \cdot (0.2K^{-0.4} + 0.8L^{-0.4})^{-2.5}$

$K_0, L_0 \rightarrow \quad Q_0 = 18 \cdot (0.2K_0^{-0.4} + 0.8L_0^{-0.4})^{-2.5}$

$tK_0, tL_0 \rightarrow \quad Q = 18 \cdot (0.2(tK_0)^{-0.4} + 0.8(tL_0)^{-0.4})^{-2.5}$

$Q = (t^{-0.4})^{-2.5} \cdot 18(0.2K_0^{-0.4} + 0.8L_0^{-0.4})^{-2.5}$

$Q = t^1 \cdot Q_0$

$\therefore$ Degree of HM = 1 which is constant return to scale. //

1.2. $\frac{dQ}{dK} = 18 \cdot (-2.5)[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5} (-0.08K^{-1.4}) > 0$

$\frac{dQ}{dL} = 18 \cdot (-2.5)[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5} (-0.32L^{-1.4}) > 0$

$\therefore$ The production function increasing with respect to K and L. //

1.3. $f(K,L) = 0$

$f(K,L) = 18 \cdot (0.2K^{-0.4} + 0.8L^{-0.4})^{-2.5} = 0$

$$MRTS = -\frac{F_L}{F_K} = \frac{\cancel{18 \cdot (-2.5)[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5}} (-0.32L^{-1.4})}{\cancel{18 \cdot (-2.5)[0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5}} (-0.08K^{-1.4})}$$

$\frac{MP_L}{MP_K}$

$$= \frac{4L^{-1.4}}{K^{-1.4}}$$ //

1.4 Use the Hessian matrix. Proof that the production function is concave.

$$Q = f(\overset{x}{K}, \overset{y}{L}) = 18 * [0.2K^{-0.4} + 0.8L^{-0.4}]^{-2.5}$$

$$H = \begin{bmatrix} f_{KK} & f_{KL} \\ f_{LK} & f_{LL} \end{bmatrix}$$

- $f_{KK} = \frac{\partial f}{\partial K} = (3.6K^{-1.4})(0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5}$

$$\frac{\partial f_K}{\partial K} = (3.6K^{-1.4})(3.5)[0.2K^{-0.4} + 0.8L^{-0.4}]^{-4.5}(-0.08K^{-1.4}) + (0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5}(-5.04K^{-2.4})$$
$$= (1.008K^{-2.8})[0.2K^{-0.4} + 0.8L^{-0.4}]^{-4.5} + [0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5}(-5.04K^{-2.4})$$

- $f_{KL} = \frac{\partial f}{\partial K} = (3.6K^{-1.4})(0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5}$

$$\frac{\partial f_K}{\partial L} = (3.6K^{-1.4})(-3.5)[0.2K^{-0.4} + 0.8L^{-0.4})^{-4.5}(-0.32L^{-1.4}) + [0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5}(0)$$
$$= (4.032K^{-1.4}L^{-1.4})[0.2K^{-0.4} + 0.8L^{-0.4}]^{-4.5}$$

- $f_{LK} = \frac{\partial f}{\partial L} = (14.4L^{-1.4})(0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5}$

$$\frac{\partial f_L}{\partial K} = (14.4L^{-1.4})(-3.5)(0.2K^{-0.4} + 0.8L^{-0.4})^{-4.5}(-0.08K^{-1.4}) + [0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5}(0)$$
$$= (4.032K^{-1.4}L^{-1.4})[0.2K^{-0.4} + 0.8L^{-0.4})^{-4.5}$$

- $f_{LL} = \frac{\partial f}{\partial L} = (14.4L^{-1.4})(0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5}$

$$\frac{\partial f_L}{\partial L} = (14.4L^{-1.4})(-3.5)(0.2K^{-0.4} + 0.8L^{-0.4})^{-4.5}(-0.32L^{-1.4}) + (0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5}(-20.16L^{-2.4})$$
$$= (16.128L^{-2.8})(0.2K^{-0.4} + 0.8L^{-0.4})^{-4.5} + [0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5}(-20.16L^{-2.4})$$

$$|H_1| = \left|(1.008K^{-2.8})[0.2K^{-0.4} + 0.8L^{-0.4}]^{-4.5} + [0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5}(-5.04K^{-2.4})\right|$$
$$= (1.008K^{-2.8})[0.2K^{-0.4} + 0.8L^{-0.4}]^{-4.5} + [0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5}(-5.04K^{-2.4})$$

$$|H_2| = \begin{bmatrix} f_{KK} & f_{KL} \\ f_{LK} & f_{LL} \end{bmatrix}$$

$$= \Big[\{(1.008K^{-2.8})[0.2K^{-0.4} + 0.8L^{-0.4}]^{-4.5} + [0.2K^{-0.4} + 0.8L^{-0.4}]^{-3.5}(-5.04K^{-2.4})\} \times \{(16.128L^{-2.8})(0.2K^{-0.4} + 0.8L^{-0.4})^{-4.5} + (0.2K^{-0.4} + 0.8L^{-0.4})^{-3.5}(-20.16L^{-2.4})\}\Big]$$
$$- \Big[\{(4.032K^{-1.4}L^{-1.4})(0.2K^{-0.4} + 0.8L^{-0.4})^{-4.5}\} \times \{(4.032K^{-1.4}L^{-1.4})(0.2K^{-0.4} + 0.8L^{-0.4})^{-4.5}\}\Big]$$

$\because |H_1| < 0 \; ; \; \forall K, \forall L$

$|H_2| > 0 \; ; \; \forall K, \forall L$

$\therefore$ H is negative definite due to $d^2Q < 0$. So, the production function is concave.

Question 2: Define f(x,y) for all (x,y) by

$$f\,(x,y)\ =\ e^{x+y} +\ e^{x-y} - \tfrac{3}{2}\,x\ -\ \tfrac{1}{2}y$$

2.1 Derive the Hessian matrix of $f\ (x,y)$.

2.2 Show that f(x,y) is monotonically convex function. That's, the function is convex for the entire domain sets defining f(x,y).

2.3 Find the *global* extrema of f(x,y). What type of extrema is it?

2.1. Derive the Hessian matrix of $f(x,y)$

$$f_x = \frac{\partial f}{\partial x} = e^{x+y} + e^{x-y} - \frac{3}{2}$$

$$f_y = \frac{\partial f}{\partial y} = e^{x+y} - e^{x-y} - \frac{1}{2}$$

$$f_{xx} = e^{x+y} + e^{x-y}$$

$$f_{xy} = e^{x+y} - e^{x-y}$$

$$f_{yx} = e^{x+y} - e^{x-y}$$

$$f_{yy} = e^{x+y} + e^{x-y}$$

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} e^{x+y} + e^{x-y} & e^{x+y} - e^{x-y} \\ e^{x+y} - e^{x-y} & e^{x+y} + e^{x-y} \end{bmatrix}$$

2.2. Show that $f(x,y)$ is monotonically convex function

$$H = \begin{bmatrix} e^{x+y} + e^{x-y} & e^{x+y} - e^{x-y} \\ e^{x+y} - e^{x-y} & e^{x+y} + e^{x-y} \end{bmatrix}$$

$$|H_1| = \left| e^{x+y} + e^{x-y} \right| = e^{x+y} + e^{x-y} > 0 \quad \forall x, \forall y$$

$$|H_2| = \begin{vmatrix} e^{x+y} + e^{x-y} & e^{x+y} - e^{x-y} \\ e^{x+y} - e^{x-y} & e^{x+y} + e^{x-y} \end{vmatrix}$$

$$= (e^{x+y} + e^{x-y})(e^{x+y} + e^{x-y}) - (e^{x+y} - e^{x-y})(e^{x+y} - e^{x-y})$$

$$= (e^{x+y} + e^{x-y})^2 - (e^{x+y} - e^{x-y})^2 > 0 \quad \forall x, \forall y$$

$|H_i| > 0 \quad \forall x, \forall y$

H is positive definite.

$d^2 f > 0$, for all (x, y)

$\Rightarrow f(x,y)$ is monotonically convex function.

2.3. Find the global extrema of $f(x,y)$

Two-steps Process

Step-1 $\Rightarrow f_x = f_y = 0$

$$f_x = \frac{\partial f}{\partial x} = e^{x+y} + e^{x-y} - \frac{3}{2} = 0 \quad (1)$$

$$f_y = \frac{\partial f}{\partial y} = e^{x+y} - e^{x-y} - \frac{1}{2} = 0 \quad (2)$$

$$e^{x+y} + e^{x-y} = \frac{3}{2} \quad (1)$$

$$e^{x+y} - e^{x-y} = \frac{1}{2} \quad (2)$$

(1) + (2): $$2e^{x+y} = 2$$

$$e^{x+y} = 1$$

$$\Rightarrow e^{x+y} = e^0$$

$$x + y = 0$$

$$x = -y$$

(1): $$e^{-y+y} + e^{-y-y} = \frac{3}{2}$$

$$e^0 + e^{-2y} = \frac{3}{2}$$

$$e^{-2y} = \frac{1}{2}$$

$$-2y = \ln\frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$$

$$\left.\begin{aligned} y &= \frac{\ln 2}{2} \\ \Rightarrow x &= -\frac{\ln 2}{2} \end{aligned}\right\} \left(-\frac{\ln 2}{2}, \frac{\ln 2}{2}\right) \Rightarrow \text{critical point}$$

Step-2: From 2.2.

all $|H_i| > 0 \quad \forall i$

H is always positive definite.

$\Rightarrow d^2 f > 0$ at the critical point.

$\Rightarrow f$ is convex at the critical point.

$\left(-\frac{\ln 2}{2}, \frac{\ln 2}{2}\right)$ is the local minimizer.

Since $f(x,y)$ is monotonically convex function,

$\left(-\frac{\ln 2}{2}, \frac{\ln 2}{2}\right)$ is the global minimizer.

$\Rightarrow$ The global minimum is :

$$f\left(-\frac{\ln 2}{2}, \frac{\ln 2}{2}\right) = e^{-\frac{\ln 2}{2}+\frac{\ln 2}{2}} + e^{-\frac{\ln 2}{2}-\frac{\ln 2}{2}} - \frac{3}{2}\left(-\frac{\ln 2}{2}\right) - \frac{1}{2}\left(\frac{\ln 2}{2}\right)$$

$$= e^{0} + e^{-\ln 2} + \frac{3\ln 2}{4} - \frac{\ln 2}{4}$$

$$= 1 + \frac{1}{e^{\ln 2}} + \frac{\ln 2}{2}$$

$$= 1 + \frac{1}{2} + \frac{\ln 2}{2} = \frac{3}{2} + \frac{\ln 2}{2}$$

Question 3:

A monopolist faces the market demand given by $P = Q^{-c}$ where "c" is a parameter with positive value, "P" is the price per unit output and "Q" is the amount of output. Suppose that monopolist's production technology is given by $Q = K^{\frac{1}{3}}L^{\frac{2}{3}}$ where "K" and "L" are the level of capital used and

the number of labor employed, respectively. Assume that the unit price of K and L are set equal to "r" and "w", respectively. Consider the following problems.

3.1) What type of the return to scale technology does the production function exhibit?

$$K_0, L_0 \rightarrow Q = K^{\frac{1}{3}}L^{\frac{2}{3}}$$

$$tK_0, tL_0 \rightarrow Q = (tK)^{\frac{1}{3}}(tL)^{\frac{2}{3}}$$

$$= t^{\frac{1}{3}}K^{\frac{1}{3}} \cdot t^{\frac{2}{3}}L^{\frac{2}{3}}$$

$$= t^{\left(\frac{3}{3}\right)}\underbrace{\left(K^{\frac{1}{3}}L^{\frac{2}{3}}\right)}_{Q}$$

degree $= 1$

Since, the degree of the homogenous fn is equal to 1, the production function exhibit a constant return to scale.

From now on, assume that c = ¼ . Consider the following problems.

3.2) Construct the profit function of the monopolist. (Hint: your profit function should be expressed in terms of K and L.)

$$\pi = Rev - cost$$

$$= PQ - wL - rk$$

$$= Q^{-c}Q - wL - rk$$

$$c = \frac{1}{4} \Rightarrow Q^{-c}Q = Q^{\frac{3}{4}} = \left(K^{\frac{1}{3}}L^{\frac{2}{3}}\right)^{\frac{3}{4}}$$

$$\pi(K, L) = \underbrace{\left[K^{\frac{1}{4}}L^{\frac{1}{2}}\right]}_{R(Q)} - wL - rk$$

3.3) The firm wants to maximize profit and seek for combination of the two factor inputs . Derive the demand for factor inputs, capital and labor.

$$\frac{\partial \pi}{\partial k} = \frac{1}{4}L^{\frac{1}{2}}K^{-\frac{3}{4}} - r = 0$$

$$K^{-\frac{3}{4}} = \frac{4r}{L^{\frac{1}{2}}} \Rightarrow K^{*} = \left(\frac{4r}{L^{\frac{1}{2}}}\right)^{-\frac{4}{3}}$$

$$\frac{\partial \pi}{\partial L} = \frac{1}{2}K^{\frac{1}{4}}L^{-\frac{1}{2}} - w = 0$$

$$L^{-\frac{1}{2}} = \frac{2w}{K^{\frac{1}{4}}}$$

$$L^{*} = \left(\frac{2w}{K^{\frac{1}{4}}}\right)^{-2}$$

3.4) How does the demand for labor vary with respect to w and r? Show your result by using partial derivative.

$$L^* = \left(\frac{2w}{k^{\frac{2}{4}}}\right)^{-2}$$

$$= \frac{(2w)^{-2}}{(k^{\frac{1}{4}})^{-2}}$$

$$= \frac{-4w^{-2}}{k^{-\frac{1}{2}}}$$

$$= -4w^{-2}k^{\frac{1}{2}}$$

$$k^* = \left(\frac{4r}{2^{\frac{1}{2}}}\right)^{-\frac{4}{3}}$$

$$k^* = \frac{4^{-\frac{4}{3}} r^{-\frac{4}{3}}}{2^{\frac{1}{2} \times -\frac{4}{3}}}$$

$$= \frac{1}{4^{\frac{4}{3}} r^{\frac{4}{3}} 2^{-\frac{2}{3}}}$$

$$k^* = \frac{1}{4r^{\frac{4}{3}}}$$

substitude k into L^*

$$L^* = -4w^{-2}(4r^{\frac{4}{3}})^{\frac{1}{2}}$$

$$= -4w^{-2}(2r^{\frac{2}{3}})$$

$$L^* = -8w^{-2}r^{\frac{2}{3}}$$

$$\frac{\partial L^*}{\partial w} = -8(-2)w^{-3}r^{\frac{2}{3}}$$

$$= -16w^{-3}r^{\frac{2}{3}}$$

∴ when the wage increase by 1 unit, the Labour would fall by $16w^{-3}r^{\frac{2}{3}}$ units

$$\frac{\partial L^*}{\partial r} = -8w^{-2}\left(\frac{2}{3}\right)r^{-\frac{1}{3}}$$

$$= -\frac{16}{3}w^{-2}r^{-\frac{1}{3}}$$

∴ when the capital increase by 1 unit, the labour would fall by $\frac{16}{3}w^{-2}r^{-\frac{1}{3}}$ units

3.5) Confirm your answer with the second-order condition.

$$\pi_K = \frac{1}{4}L^{\frac{1}{2}}K^{-\frac{3}{4}} - r \quad , \quad \pi_L = \frac{1}{2}K^{\frac{1}{4}}L^{-\frac{1}{2}} - w$$

$$H = \begin{bmatrix} \pi_{KK} & \pi_{KL} \\ \pi_{LK} & \pi_{LL} \end{bmatrix} ; \quad \pi_{KK} = -\frac{3}{16}K^{-\frac{7}{4}}L^{\frac{1}{2}} \quad \pi_{KL} = \frac{1}{8}K^{-\frac{3}{4}}L^{-\frac{1}{2}}$$

$$\pi_{LK} = \frac{1}{8}K^{-\frac{3}{4}}L^{-\frac{1}{2}} \qquad \pi_{LL} = -\frac{1}{4}K^{\frac{1}{4}}K^{-\frac{3}{2}}$$

$$|H_2| = \begin{bmatrix} -\frac{3}{16}K^{-\frac{7}{4}}L^{\frac{1}{2}} & \frac{1}{8}K^{-\frac{3}{4}}L^{-\frac{1}{2}} \\ \frac{1}{8}K^{-\frac{3}{4}}L^{-\frac{1}{2}} & -\frac{1}{4}K^{\frac{1}{4}}L^{-\frac{3}{2}} \end{bmatrix}$$

$$= \left(-\frac{3}{16}K^{-\frac{7}{4}}L^{\frac{1}{2}}\right)\left(-\frac{1}{4}K^{\frac{1}{4}}L^{-\frac{3}{2}}\right) - \left(\frac{1}{8}K^{-\frac{3}{4}}L^{-\frac{1}{2}}\right)\left(\frac{1}{8}K^{-\frac{3}{4}}L^{-\frac{1}{2}}\right)$$

$$= \frac{3}{64}K^{-\frac{3}{2}}L^{-1} - \frac{1}{64}K^{-\frac{3}{2}}L^{-1}$$

$$= \frac{1}{32}K^{-\frac{3}{2}}L^{-1}$$

$$|H_1| = -\frac{3}{16}K^{-\frac{7}{4}}L^{\frac{1}{2}}$$

$$\therefore \left.\begin{matrix} H_1 < 0 \\ H_2 > 0 \end{matrix}\right\} \begin{matrix}\text{it mean that it is negative definite} \\ d^2\pi \text{ is always zero. So, it is globally concave.}\end{matrix}$$