Quiz 2: Date: May 5, 2022 from 11.00-12.30

Question 1 (40 marks) Score.....................

Consider the Muliperiod model of consumption and portfolio choice. Let an individual in this economy has the utility function as follow:

$$\max_{C_s, \omega_s, \forall_t} E_t \left[\sum_{s=t}^{T-1} \delta^s \left(\frac{C_s^{(1-\gamma)}}{1-\gamma} \right) + \delta^T \left(\frac{W_T^{(1-\gamma)}}{1-\gamma} \right) \right]$$

Assume that there is no wage income $(y_t = 0 \, \forall t)$ and a constant risk-free rate return asset , $R_{ft} = R_f$. Also asssume that n=1 and the return of a single risky asset, R_{rt},is independently and identically distributed over time. Denote the proportion of wealth invested in the risky asset at date t as ω_t.

Please read and answer the following questions carefully and completely.

Score.....................

Question 1.1 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-1, C^*_{T-1} and ω^*_{T-1}., and give an explicit expression for C^*_{T-1}

Let $S_t = W_t - C_t$ $\qquad U(C_t, t) = \delta^t \frac{C_t^{(1-\gamma)}}{1-\gamma}$

$R_t = R_f + w_t^*(R_H - R_f)$ $\qquad B(U_T, T) = \delta^T \frac{W_T^{(1-\gamma)}}{1-\gamma}$

$$U_c = E_{T-1}[B_w R_{T-1}]$$

$$\delta^{T-1} C_{T-1}^{-\gamma} = E_{T-1}[\delta^T W_T^{-\gamma} R_{T-1}]$$

$$= E_{T-1}[\delta^T (S_{T-1} R_{T-1})^{-\gamma} R_{T-1}]$$

$$= \delta^T E_{T-1}[R_{T-1}^{1-\gamma}] (W_{T-1} - C_{T-1})^{-\gamma}$$

Rearrange ; $$C^*_{T-1} = \frac{(\delta E_{T-1}[R_{T-1}^{1-\gamma}])^{-\frac{1}{\gamma}}}{1 + (\delta E_{T-1}[R_{T-1}^{1-\gamma}])^{-\frac{1}{\gamma}}} W_{T-1}$$

$$= \frac{a_1}{1+a_1} W_{T-1} = c_1 W_{T-1}$$

let $a_1 = (\delta E_{T-1}[R_{T-1}^{1-\gamma}])^{-\frac{1}{\gamma}}$ & $c_1 = \frac{a_1}{1+a_1}$

$= (\delta E_T[R_{T-1}^{1-\gamma}])^{-\frac{1}{\gamma}}$

$$E_{T-1}[B_w R_{r,T-1}] = R_f E_{T-1}[B_w]$$

$$\delta^T E_{T-1}[[S_{T-1} R_{T-1}]^{-\gamma} R_{r,T-1}] = \delta^T R_{f,T-1} E_{T-1}[(S_{T-1} R_{T-1})^{-\gamma}]$$

$$E_{T-1}[R_{T-1}^{-\gamma} R_{r,T-1}] = R_{f,T-1} E_{T-1}[R_{T-1}^{-\gamma}]$$

∴ Weight is independent from level of wealth or consumption.

Score.....................

Question 1.2 (10 marks) Solve for the form of J(W_{T-1},T-1).

let $a_1 = \left(\delta E_{T-1}[R_{T-1}^{1-\gamma}]\right)^{-\frac{1}{\gamma}}$

$= \left(\delta E_T[R_{T-1}^{1-\gamma}]\right)^{-\frac{1}{\gamma}}$

$$J(W_{T-1},T-1) = \delta^{T-1}\left(\frac{C_{T-1}^{(1-\gamma)}}{1-\gamma}\right) + \delta^T E_{T-1}\left[\frac{(R_{T-1}(W_{T-1}-C_{T-1}))^{(1-\gamma)}}{1-\gamma}\right]$$

$$= \delta^{T-1}\left(\frac{a_1}{1+a_1}\right)^{(1-\gamma)}\frac{W_{T-1}}{1-\gamma} + \delta^T E_{T-1}\left[\frac{R_{T-1}^{(1-\gamma)}\left(W_{T-1}-\frac{a_1}{1+a_1}W_{T-1}\right)^{(1-\gamma)}}{1-\gamma}\right]$$

$$= \delta^{T-1}\left(\frac{a_1}{1+a_1}\right)^{(1-\gamma)}\frac{W_{T-1}}{1-\gamma} + \delta^T E_{T-1}\left[R_{T-1}^{(1-\gamma)}\frac{W_{T-1}^{(1-\gamma)}}{(1-\gamma)(1+a_1)^{(1-\gamma)}}\right]$$

$$= \delta^{T-1}\frac{W_{T-1}^{1-\gamma}}{(1-\gamma)(1+a_1)^{1-\gamma}}\left(a_1^{1-\gamma} + \delta E_{T-1}[R_{T-1}^{(1-\gamma)}]\right)$$

$$\text{Let } b_1 \equiv \left(a_1^{1-\gamma} + \delta E[R_{T-1}^{(1-\gamma)}]\right)/(1+a_1)^{1-\gamma} = \left[a_1^{1-\gamma} + a_1^{-\gamma}\right]/(1+a_1)^{1-\gamma})$$

$$= (a_1 a_1^{-\gamma} + a_1^{-\gamma})/(1+a_1)^{1-\gamma}$$

$$= \frac{a_1^{-\gamma}(a_1+1)}{(1+a_1)^{1-\gamma}}$$

$$= \frac{(1+a_1)^{\gamma}}{a_1^{\gamma}}$$

$$= \left(\frac{a_1}{1+a_1}\right)^{-\gamma}$$

$a_1 = \left(\delta E_T[R_{T-1}^{1-\gamma}]\right)^{-\frac{1}{\gamma}}$

$a_1^{-\gamma} = \delta E_T[R_{T-1}^{1-\gamma}]$

Score.....................

Let $b_1 = \left(\frac{a_1}{1+a_1}\right)^{-\gamma}$ From 1.2

Question 1.3 (10 marks) Derive the first-order condition for the optimal consumption level and portfolio weight at date T-2, C^*_{T-2} and ω^*_{T-2}., and give an explicit expression for C^*_{T-2}

$$V_c(C^*_{T-2}, T-2) = E_{T-2}[J_w(W_{T-1}, T-1)R_{T-2}]$$

$$C_{T-2}^{-\gamma} = \delta^{T-1} E_{T-2}[b_1 W_{T-1}^{-\gamma} R_{T-2}]$$

$$C_{T-2}^{-\gamma} = \delta E_{T-2}[b_1 (S_{T-2}R_{T-2})^{-\gamma} R_{T-2}]$$

$$= \delta b_1 E_{T-2}[R_{T-2}^{1-\gamma}](W_{T-2} - C_{T-2})^{-\gamma}$$

$$C^*_{T-2} = \frac{(\delta b_1 E_{T-2}[R_{T-2}^{1-\gamma}])^{-\frac{1}{\gamma}}}{1 + (\delta b_1 E_{T-1}[R_{T-2}^{1-\gamma}])^{-\frac{1}{\gamma}}} W_{T-2}$$

$$= \frac{a_2}{1+a_2} W_{T-2} = c_2 W_{T-2}$$

$a_2 \equiv (b_1 \delta E[R_{T-2}^{1-\gamma}])^{-\frac{1}{\gamma}}$, $c_2 = a_2/(1+a_2)$

$$E_{T-2}[J_W R_{r,T-2}] = R_f E_{T-2}[J_W]$$

$$E_{T-2}[b_1(S_{T-2}R_{T-2})^{-\gamma} R_{r,T-2}] = R_f E_{T-1}[b_1(S_{T-2}R_{T-2})^{-\gamma}]$$

$$E_{T-2}[b_1 R_{T-2}^{-\gamma} R_{r,T-2}] = R_f E_{T-2}[b_1 R_{T-2}^{-\gamma}]$$

$$E[R_{T-2}^{\gamma-1} R_{r,T-2}] = R_f E[R_{T-2}^{-\gamma}]$$

$\therefore$ w^*_{T-2} is the same as w^*_{T-1}

Score......................

Question 1.4 (10 marks) Solve for the form of J(W_{T-2},T-2).Based on the pattern for T-1 and T-2, provide expressions for the optimal consumption and portfolio weight at any date T-t, t=1,2,3,....

$$J(W_{T-2}, T-2) = U(C^*_{T-2}, T-2) + E_{T-1}[J(W_{T-1}, T-1)]$$

$$= \delta^{T-2} C^{1-\gamma}/1-\gamma + E_{T-1}\left[\delta^{T-1} b_1 W^{1-\gamma}_{T-1} / 1-\gamma\right]$$

$$= \delta^{T-2}\left(\frac{a_2}{1+a_2}\right)^{1-\gamma} W^{1-\gamma}_{T-2}/_{1-\gamma} + \delta^{T-1} E_{T-1}\left[b_1 R^{*\,1-\gamma}_{T-2} \frac{W^{1-\gamma}_{T-2}}{(1-\gamma)(1+a_2)^{1-\gamma}}\right]$$

$$= \delta^{T-2} \frac{W^{1-\gamma}_{T-2}}{(1-\gamma)(1+a_2)}\left(a_2^{1-\gamma} + \delta E_{T-2}[b_1 R^{*(1-\gamma)}_{T-2}]\right)$$

$$= \delta^{T-1} b_2 W^{1-\gamma}_{T-2}/1-\gamma$$

where $b_2 = \left[a_2^{(1-\gamma)} + \delta E[R^{*\gamma}_{T-2}]\right) / (1+a_2)^{(1-\gamma)}$

The level of consumption at T-1 is $C^*_{T-2} = c_2 W_{T-2}$ where $c_2 = a_1 c_1 / (1 + a_1 c_1)$

and same for c^*_{T-3} . . . . In general, $C^*_{T-t} = c_t W_{T-t}$

Where $c_t = a_1 c_{t-1} / (1 + a_1 c_{t-1})$