Exercise 3 (Part 2)

1. (a) Prove the statement:
"There are distinct integers m and n such that $\frac{1}{m} + \frac{3}{n}$ is an integer. "

(b) Disprove the statement: " For all integers x and y, if $3x+y$ is odd then x and y are both odd. "

2. Show that "for all integers m and n, if mn is even, then m is even or n is even," by using
a) a proof by contraposition,
b) a proof by contradiction.

3. Prove by the **method of exhaustion** that " $n^2 + 1 \geq 2^n$ for any positive integer n with $1 \leq n \leq 4$."

4. Use the **proof by cases** to show that " Prove that for all integers m and n,
$m + n$ and $m - n$ are either both odd or both even."

[Hint: Consider 4 cases of even and odd for m and n]

5. Consider the statement: for any integer $n \geq 0$,

$$2 + 2 \cdot 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + \cdots + 2 \cdot 5^n = \frac{1}{2}(5^{n+1} - 1).$$

Suppose we want to prove the above statement by **mathematical induction**.

(a) What is $P(n)$?
(b) Write $P(0)$: Is $P(0)$ true?
(c) Write $P(k)$:
(d) Write $P(k+1)$:
(e) Prove the above statement: $\sum_{j=0}^{n} 2 \cdot 5^j = \frac{1}{2}(5^{n+1} - 1)$ by using **mathematical induction**.

6. Use mathematical induction proof to show that

$$2^n > n^2$$

for an integer greater than 4.

7. (Optional) Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.

8. (Optional) Use the method of constructive proof to show that:
if r and s are two real numbers with $r < s$ then there exists a real number x such that $r < x < s$.

9. (Optional) Prove by contradiction that the difference of any rational number and any irrational number is irrational.

10. (Optional) A sequence $a_1, a_2, \ldots$ is defined recursively by

$$a_1 = 3, \quad a_i = 7a_{i-1} \quad \text{for} \quad i \geq 2.$$

Show that

$$a_n = 3 \cdot 7^{n-1} \quad \text{for} \quad n \geq 1.$$