ASSIGNMENT KEY #1.

Name ________ Surname ________ Student ID. ________

DUE DATE : Tuesday 9^{th}, February 2016.

Assignment 1: (10 marks)

I pledge to the Honor Code and to obey all rules for taking and performing homework assignments as specified by the course instructor.

Student Signature: ________

1. Let X and Y have the join probability density function specified in the following table.

		X			
		0	1	2	
	0	0	0.20	0.10	$f(Y=0)$ $= 0.3$
Y	1	0.20	0.40	0	$f(Y=1)$ $= 0.6$
	2	0.10	0	0	$f(Y=2)$ $= 0.1$
		$f(X=0)$ $= 0.3$	$f(X=1)$ $= 0.6$	$f(X=2)$ $= 0.1$	

[1]. Find $E(X^2|Y=2)$ and $var(X|Y=2)$

$E(X^2|Y=2) = \sum_x X^2 f(X|Y=2)$

$= 0^2 f(X=0|Y=2) + 1^2 f(X=1|Y=2) + 2^2 f(X=2|Y=2)$

$= 0^2 \frac{f(X=0, Y=2)}{f(Y=2)} + 1^2 \frac{f(X=1, Y=2)}{f(Y=2)} + 2^2 \frac{f(X=2, Y=2)}{f(Y=2)}$

$= 0 + \frac{1(0)}{0.1} + \frac{2(0)}{0.1} = 0$ #.

$var(X|Y=2) = E[X - E(X|Y=2)]^2$

• $E(X|Y=2) = 0 \cdot f(X=0|Y=2) + 1 \cdot f(X=1|Y=2) + 2 \cdot f(X=2|Y=2) = 0$ #

$var(X|Y=2) = (0-0)^2 f(X=0|Y=2) + (0-1)^2 f(X=1|Y=2) + (0-2)^2 f(X=2|Y=2)$

$= 0$ #

$-x^2$), if $-1 <$
otherwise.

2. Suppose the joint distribution of X and Y is such that

$E[X] = 5 \qquad E[X^2] = 30 \qquad E[Y|X] = 3X \qquad E[XY|X] = 3X^2 \qquad Var[Y] = 81$

Find (a) Cov(X,Y) and (b) Corr(X,Y)

$Cov(X,Y) = E(XY) - E(X)E(Y)$

since $E(Y) = E_x[E(Y|X)]$ by the properties

$E_x(3X) = 3E(X) = 3(5) = 15$ #

$E(XY) = E[E(XY|X)] = E(XE(Y|X)) = E(X(3X))$

$= E(3X^2) = 3E(X^2)$

$= 3(30) = 90$ #

$\therefore Cov(X,Y) = E(XY) - E(X)E(Y)$

$= 90 - 5(15) = 90 - 75 = \boxed{15}$ #

$Corr(X,Y) = \dfrac{Cov(X,Y)}{\sigma_x \sigma_y}$

$\sigma_x = \sqrt{var(X)} = \sqrt{5}$

$\sigma_y = \sqrt{var(Y)} = \sqrt{81} = 9$

$var(X) = E(X^2) - [E(X)]^2$

$= 30 - 5^2 = 5$

$\therefore \rho_{xy} = \dfrac{15}{\sqrt{5} \cdot 9} = 0.745$ #

3. Let X and Y be random variables with the following joint distribution

$$f(x, y) = P(X = x, Y = y) = 1/8$$

If x $\in \{-1, 0, 1\}$ and y $\in \{-1, 0, 1\}$ BUT (x,y) $\neq$ (0,0)

Show that the Cov(X,Y) =0 BUT X and Y ARE NOT INDEPENDENT.

		X			
		-1	0	1	
Y	-1	1/8	1/8	1/8	$f(y=-1) = 3/8$
	0	1/8	0	1/8	$f(y=0) = 1/4$
	1	1/8	1/8	1/8	$f(y=1) = 3/8$
		$f(x=-1)=\frac{3}{8}$	$f(x=0)=\frac{1}{4}$	$f(x=1)=\frac{3}{8}$	

$COV(X,Y) = E(XY) - E(X)E(Y)$

$E(XY) = \sum_y \sum_x x_i y_i f(x_i, y_i)$

$= -1(-1)(\frac{1}{8}) + 0(-1)(\frac{1}{8}) + 1(-1)(\frac{1}{8}) + (-1)(0)(\frac{1}{8}) + (0)(0)(0) + 1(0)(\frac{1}{8}) + (-1)(1)(\frac{1}{8}) + 0(1)(\frac{1}{8}) + (1)(1)(\frac{1}{8})$

$= \frac{1}{8} - \frac{1}{8} + \frac{1}{8} - \frac{1}{8} = 0$

$E(X) = \sum_{i=1}^{3} x_i f(x_i)$

$= -1(\frac{3}{8}) + 0(\frac{1}{4}) + 1(\frac{3}{4}) = 0$

$E(Y) = \sum_{i=1}^{3} y_i f(y_i)$

$= -1(\frac{3}{8}) + 0(\frac{1}{4}) + 1(\frac{3}{8}) = 0$

$\therefore Cov(X,Y) = 0 - 0 = 0$ #

But when $X = -1$ and $y = -1$

$f(x,y) = \frac{1}{8}$, $f(x=-1) = \frac{3}{8}$, $f(y=-1) = \frac{3}{8}$

$f(x=-1)f(y=-1) = \frac{3}{8} \cdot \frac{3}{8} = \frac{9}{64} \neq f(x,y) = \frac{1}{8}$

therefore, $cov(x,y) = 0$ but x and y are not independent #

4. Show the derivation if X and Y are two **independent random variables**, then

$$var(X+Y) = var(X) + var(Y)$$

$var(X+Y) = var(X) + var(Y) + 2Cov(X,Y)$

when x and y are independent random variables

$Cov(X,Y) = E(XY) - \mu_X \mu_Y$ → $E(XY) = E(X)E(Y)$

$= E(X) \cdot E(Y) - \mu_X \mu_Y$ "property of independent random variable".

$= \mu_X \mu_Y - \mu_X \mu_Y$

$= 0.$ #

therefore, $var(X+Y) = var(X) + var(Y) + 0$

$\boxed{var(X+Y) = var(X) + var(Y)}$ #

5. If X_1, X_2, X_3 is a random sample from a population with mean μ and variance σ^2. X_1, X_2, X_3 **ARE NOT independent** $cov(X_1, X_2) = cov(X_1, X_3) = cov(X_2, X_3) = \frac{1}{3}\sigma^2$ and $\bar{X}$ is an estimator used to estimate mean value where $\bar{X} = \frac{1}{3}(X_1 + X_2 + X_3)$.
Find $E(\bar{X})$ and $var(\bar{X})$

$$E(\bar{X}) = E\left(\frac{1}{3}(X_1 + X_2 + X_3)\right)$$

$$= \frac{1}{3}\left[\underbrace{E(X_1)}_{\mu} + \underbrace{E(X_2)}_{\mu} + \underbrace{E(X_3)}_{\mu}\right]$$

$$= \frac{1}{3}(\mu + \mu + \mu) \ \#$$

$$= \mu \ \#$$

$$var(\bar{X}) = var\left(\frac{1}{3}(X_1 + X_2 + X_3)\right)$$

$$= \frac{1}{9} var(X_1 + X_2 + X_3)$$

$$= \frac{1}{9}\left[var(X_1) + var(X_2) + var(X_3) + 2cov(X_1, X_2) + 2cov(X_2, X_3) + 2cov(X_1, X_3)\right]$$

$$= \frac{1}{9}\left[\underbrace{var(X_1)}_{\sigma^2} + \underbrace{var(X_2)}_{\sigma^2} + \underbrace{var(X_3)}_{\sigma^2} + 2\left(\frac{1}{3}\sigma^2\right) + 2\left(\frac{1}{3}\sigma^2\right) + 2\left(\frac{1}{3}\sigma^2\right)\right]$$

$$= \frac{1}{9}\left(\sigma^2 + \sigma^2 + \sigma^2 + \frac{\cancel{6}^{2}\sigma^2}{\cancel{3}_{1}}\right) \ \#$$

$$= \frac{5}{9}\sigma^2 \ \#.$$