Chapter 1 Limits and Continuity

Section 1.1 Limits

Consider the rational function $f(x)=\dfrac{x^2-1}{x-1}$. Clearly we cannot use x = 1.

We want to know what happens if our x values approach x =1. In this case

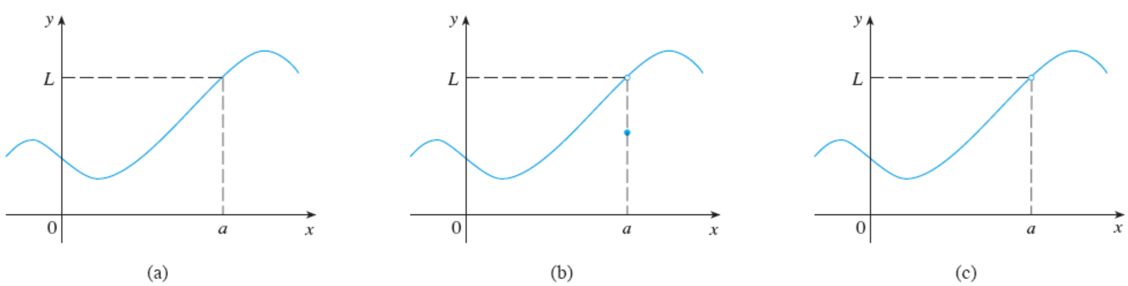
$x<1$	$f(x)$	$x>1$	$f(x)$
0.9	1.9	1.01	2.01
0.99	1.99	1.001	2.001
0.999	1.999	1.0001	2.0001

From the tables, we can see that if our x values approach x = 1, then our $f(x)$ (or y) values approach 2. This idea is expressed using the limit. We write $\lim_{x\to 1} f(x)=2$ and we read "The limit of $f(x)$ as x approaches one is two".

In general,

$$\lim_{x\to a} f(x)=L$$

In other words, we can make our y values as close to L as we wish, simply by choosing x values close enough to a.



In all of these cases, $\lim_{x\to a} f(x)=L$.

▪ One sided Limits

Consider the split definition function

$$f(x)=\begin{cases} x^2-1 & if \quad x\ge 0 \\ \frac{1}{2}x+2 & if \quad x<0 \end{cases}$$

Find $\lim_{x\to 0} f(x)$.

We have $\lim_{x\to 0} f(x)$ does not exist.

However,

$$\lim_{x\to 0^+} f(x)=-1 \quad \text{and} \quad \lim_{x\to 0^-} f(x)=2\,.$$

We write

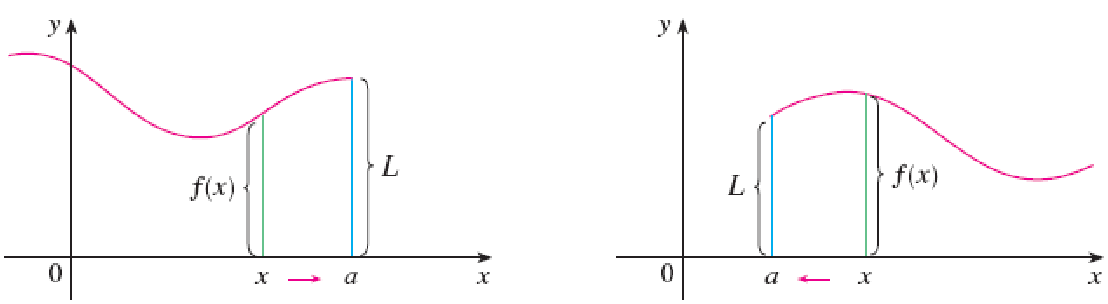
$$\lim_{x\to a^-} f(x)=L$$

"The left – hand limit of $f(x)$ as x approaches a (OR the limit of $f(x)$ as x approaches a from the left) is equal to L." If we can make the value of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x less than a.

Similarly, we write

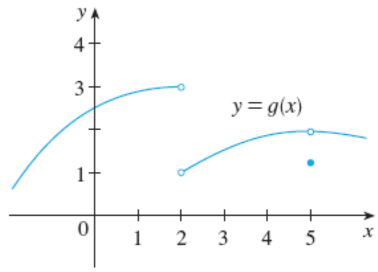
$$\lim_{x\to a^+} f(x)=L$$

"The right – hand limit of $f(x)$ as x approaches a (OR the limit of $f(x)$ as x approaches a from the right) is equal to L." If we can make the value of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a and x greater than a.



(a) $\lim_{x\to a^-} f(x)=L$ (b) $\lim_{x\to a^+} f(x)=L$

Example 1: Given $g(x)$



Find

(a) $\lim_{x\to 2^-} g(x)$ (b) $\lim_{x\to 2^+} g(x)$ (c) $\lim_{x\to 2} g(x)$

(d) $\lim_{x\to 5^-} g(x)$ (e) $\lim_{x\to 5^+} g(x)$ (f) $\lim_{x\to 5} g(x)$

(g) $g(5)$

Example 2: Graph of $g(x)$ is given as in Example 1. Determine the values of a for which $\lim_{x\to a} g(x)$ exists.

Example 3: Sketch the graph of an example of a function f that satisfies all of the given conditions.

$\lim_{x\to 3^+} f(x) = 4$ $\lim_{x\to 3^-} f(x) = 2$ $\lim_{x\to -2} f(x) = 2$

$f(3) = 3$ $f(-2) = 1$

Section 1.2 Computing Limits

Limit Laws: Suppose that *c* is a constant and the limits

$$\lim_{x\to a} f(x) \quad \text{and} \quad \lim_{x\to a} g(x)$$

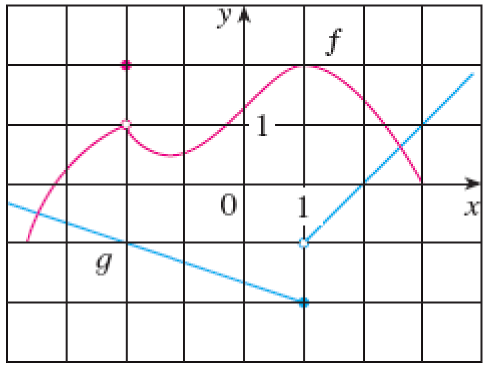
exist. Then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ where $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$ where *n* is a positive integer
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$ where *n* is a positive integer
10. $\lim_{x\to a}\sqrt[n]{x} = \sqrt[n]{a}$ If *n* is even, we assume that $a > 0$
11. $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ If *n* is even, we assume that $\lim_{x\to a} f(x) > 0$

Direct Substitution Property: If *f* is a polynomial or rational function and *a* is in the domain of *f* , then

$$\lim_{x\to a} f(x) = f(a)$$

Example 1:



Use the Limit Laws and the graphs of *f* and *g* to evaluate the following limits, if they exist.

$(a)\ \lim_{x\to -2}\left[f(x)+5g(x)\right]$

$(b)\ \lim_{x\to 1}\left[f(x)g(x)\right]$

$(c)\ \lim_{x\to 2}\dfrac{f(x)}{g(x)}$

Example 2: Find the following limits.

$$(a)\lim_{x\to 1}\frac{3x+\sqrt{x+3}}{x-2}$$

$$(b)\lim_{x\to 2}\frac{2x-8+x^2}{x-2}$$

$$(c)\lim_{x\to -1} x+5-\frac{(x+1)^2}{x^2-x-2}$$

$$(d)\lim_{x\to 2}\frac{\sqrt{x-1}-1}{x-2}$$

Note: Some limits are best calculated by first finding the left – and right – hand limits.

Theorem:

$$\lim_{x\to a} f(x) = L \quad \text{if and only if} \quad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

Example 3: Prove that $\lim_{x\to 0}\frac{|x|}{x}$ does not exist.

Example 4: Find the limit, if it exists.

$(a)\lim_{x\to 2}\frac{|x-2|}{x-2}$

$(b)\lim_{x\to 1.5}\frac{2x^2-3x}{|2x-3|}$

Example 5: If

$$f(x)=\begin{cases}\sqrt{x+6} & if\ \ x>3\\ x^2 & if\ \ x\le 3\end{cases}$$

Determine whether $\lim\limits_{x\to 3} f(x)$ exists.

Theorem: If $f(x) \le g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a , then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$$

The Squeeze Theorem: If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

then

$$\lim_{x\to a} g(x) = L\ .$$

Example 6: If $1 \le f(x) \le x^2+2x+2$ for all x , find $\lim\limits_{x\to -1} f(x)$.

Example 7: Prove that $\lim\limits_{x\to 0} x^2\cos\dfrac{2}{x} = 0$

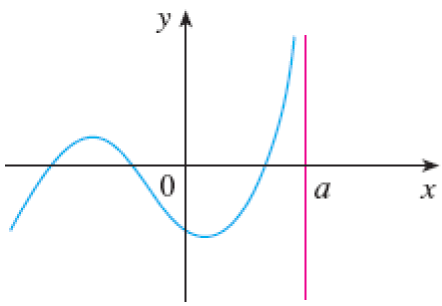
Section 1.3 Limits at Infinity

- **Vertical Asymptote**

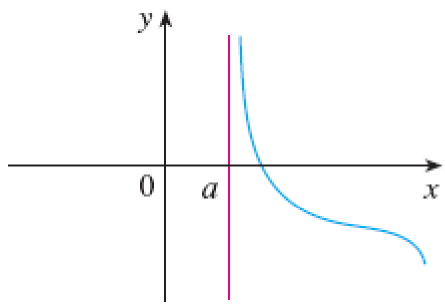
Definition: The line $x = a$ is called a *vertical asymptote* of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty$$
$$\lim_{x\to a} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty$$

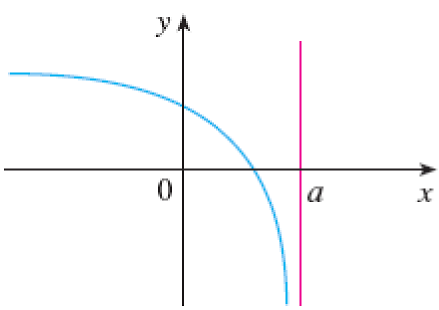
For example,



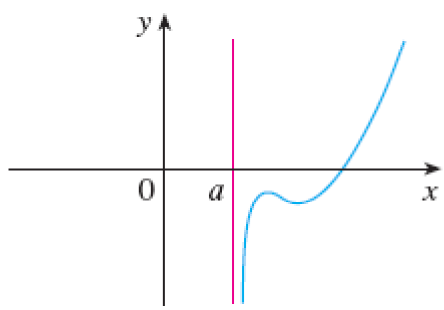
(a) $\lim_{x\to a^-} f(x) = \infty$



(b) $\lim_{x\to a^+} f(x) = \infty$



(c) $\lim_{x\to a^-} f(x) = -\infty$



(d) $\lim_{x\to a^+} f(x) = -\infty$

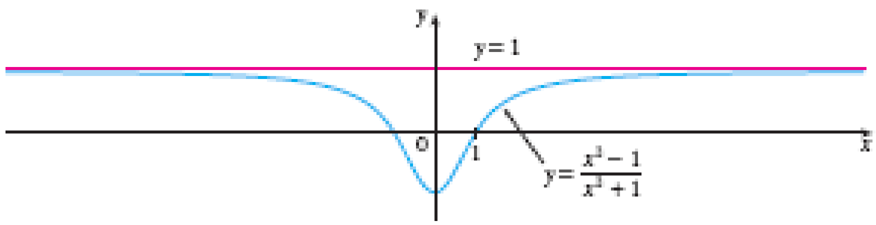
▪ Horizontal Asymptote

Definition: The line $y = L$ is called a *horizontal asymptote* of the curve $y = f(x)$ if either

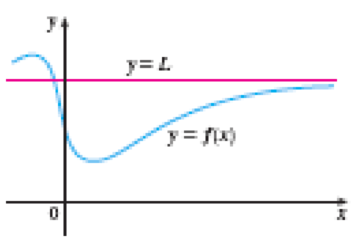
$$\lim_{x\to\infty} f(x) = L \qquad \lim_{x\to-\infty} f(x) = L$$

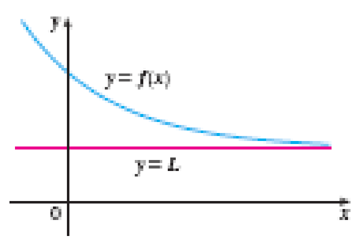
Example: Consider $f(x) = \dfrac{x^2-1}{x^2+1}$

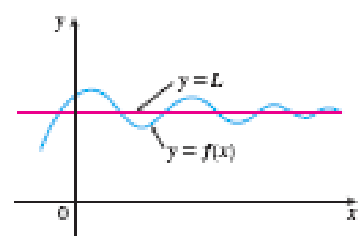
x	0	±1	±2	±3	±4	±10	±50	±100	±1000
$f(x)$	-1	0	0.6	0.8	0.882	0.98	0.9992	0.9998	0.999998



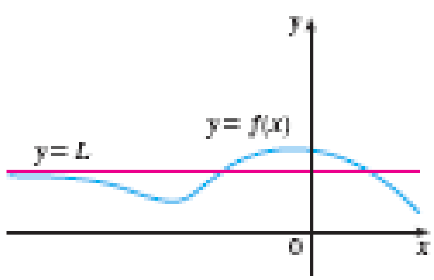
Examples illustrating $\lim_{x\to\infty} f(x) = L$:

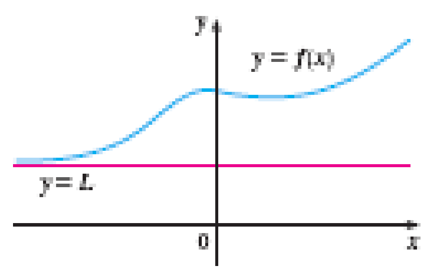






Examples illustrating $\lim_{x\to-\infty} f(x) = L$:





Theorem: If $r > 0$ is a rational number, then

$$\lim_{x\to\infty}\frac{1}{x^r}=0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty}\frac{1}{x^r}=0$$

Example 2: Find the horizontal asymptotes and vertical asymptotes of the graph of the function.

(1) $f(x)=\dfrac{x^2-x}{x^2-1}$

(2) $f(x)=\dfrac{x^2-x}{x-1}$

▪ Infinite Limits at Infinity

The notation

$$\lim_{x\to\infty} f(x) = \infty$$

is used to indicate that the values of f(x) become large. Similar meaning are attached to

$$\lim_{x\to\infty} f(x) = -\infty \quad \lim_{x\to-\infty} f(x) = \infty \quad \lim_{x\to-\infty} f(x) = -\infty$$

Example: Find

(1) $\lim_{x\to\infty} 2x^3$

(2) $\lim_{x\to-\infty} 2x^3$

(3) $\lim_{x\to-\infty} 3x^2$

(4) $\lim_{x\to\infty} \frac{x^2 - x}{x-1}$

(5) $\lim_{x\to-\infty} \frac{x^2 + x}{x+3}$

(6) $\lim_{x\to\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1}$

(7) $\lim_{x\to-\infty} \left(x + \sqrt{x^2 + 2x}\right)$

(8) $\lim_{x\to-\infty} \frac{x}{\sqrt{x^2 + 4}}$

(9) $\lim_{x\to-\infty} \left(x^4 + x^5\right)$

Section 1.4 Limits of Trigonometric Functions

Lemma1: $\lim_{x\to 0}\sin x = 0$

Lemma2: $\lim_{x\to 0}\frac{\sin x}{x} = 1$

Lemma3: For any real number k ≠ 0, $\lim_{x\to 0}\frac{\sin kx}{x} = k$

Note : From Lemma 2 and Lemma 3, we have

$$\lim_{x\to 0}\frac{x}{\sin x} = 1 \quad \text{and} \quad \lim_{x\to 0}\frac{x}{\sin kx} = \frac{1}{k}$$

Example 1: Find the following limits.

(1) $\lim_{u\to 0}\frac{\sin(3u^2)}{u^2}$

(2) $\lim_{h\to 0}\frac{\sin(2h)}{\tan(h)}$

(3) $\lim_{x\to 0}\frac{2x^2+\sin^2(3x)}{x^2}$

Section 1.5 Continuity

Notice that the limit of a function as x approaches a can often be found simply by calculating the value of the function at a. Functions with this property are called continuous at a.

Definition: A function is said to be continuous at a if

(1) $f(a)$ is defined (that is, a is in the domain of f)
(2) $\lim_{x\to a} f(x)$ exists.
(3) $\lim_{x\to a} f(x) = f(a)$.

Example:

Example 1: At which number is f discontinuous? Why?

Example 2: Given

$$f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1} & if \;\; x \neq 1 \\ 2 & if \;\; x = 1 \end{cases}$$

Is *f* continuous at 1?

Example 3: Given

$$f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1} & if \;\; x > 1 \\ x & if \;\; x \leq 1 \end{cases}$$

Is *f* continuous at 1?

Example 4: Where are each of the following functions discontinuous?

$$(a)\ f(x) = \frac{x^2 + x - 2}{x - 1}$$

$$(b)\ f(x) = \begin{cases} \dfrac{x^2 + x - 2}{x - 1} & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$$

Example 5: Find the value of *c* that makes *f* continuous on (- ∞,∞).

$$f(x) = \begin{cases} cx + 1 & \text{if } x \leq 3 \\ cx^2 - 1 & \text{if } x > 3 \end{cases}$$

Theorem: If *f* and g are continuous at *a* and *c* is *a* constant, then the following functions are also continuous at *a*:

1. $f + g$
2. $f - g$
3. cf
4. $f\,g$
5. f / g if $g(a) \neq 0$

Theorem: The following types of functions are continuous at every number in their domains:

- Polynomials
- Root functions
- Rational functions
- Trigonometric functions

Example 7: On what intervals is each function continuous?

(1) $f(x) = x^2 - 2x + 1$

(2) $g(x) = \dfrac{5x}{x+1}$

(3) $f(x) = \sqrt{x} + \dfrac{x+1}{x-1} - \dfrac{x+1}{x^2+1}$

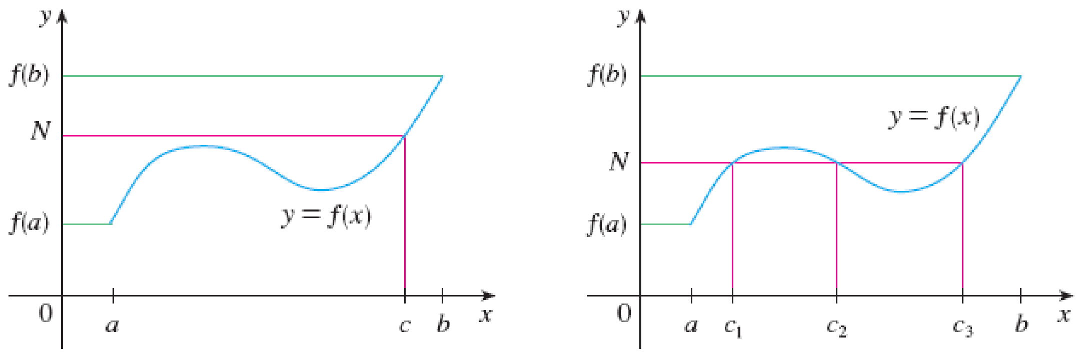
Example 8: Given

$$f(x)=\begin{cases} 1+x^2 & if\ \ x\le 0 \\ 2-x & if\ \ 0<x\le 2 \\ (x-2)^2 & if\ \ x>2 \end{cases}$$

Find the numbers at which *f* is discontinuous.

The Intermediate Value Theorem

Suppose that f is continuous on the closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a,b) such that $f(c) = N$.



(a) (b)

Example 10: Use the Intermediate Value Theorem to show that there is a root of the equation $x^4 + x - 3 = 0$ between 1 and 2.