TU 152

Example 1: Prove that if $u \vee (r \to t)$ and $\sim u \wedge \sim t$, then $\sim r$.

Example 2: Prove that if $x \to (y \to z)$, $\sim y \to \sim x$, and x, then z.

Example 3: Prove that if $\sim p \to \sim q$, $\sim u$, $p \to t$, and $q \vee u$, then t.

Example 4: Show $\sim t$ if $[r \to (s \to \sim t)] \wedge [(\sim s \to \sim r) \wedge r]$.

Example 1

1.	$\sim u \wedge \sim t$	Given
2.	$\sim u$ and $\sim t$	1. and Def^{n}. of Conjunction
3.	$u \vee (r \to t)$	Given
4.	$r \to t$	2, 3, and Disjunctive Syllogism
5	$\sim r$	2, 4, and Rule of Modus Tallens

Example 2

1.	x	Given
2.	$x \to (y \to z)$	Given
3.	$y \to z$	1, 2, Rule of Modus Ponens
4.	$x \leftrightarrow \sim(\sim x)$	Law of Double Negation
5.	$\sim(\sim x)$	1, 4, Rule of Substitution
6.	$\sim y \to \sim x$	Given
7.	$\sim(\sim y)$	5, 6, Rule of Modus Tollens
8.	$\sim(\sim y) \leftrightarrow y$	Law of Double Negation
9.	y	7, 8, Rule of Substitution
10.	z	3, 9, Rule of Modus Ponens

3) Examples 3

1. $q \vee u$ Given
2. $\sim u$ Given
3. q 1, 2, Disjunctive Syllogism
4. $q \leftrightarrow \sim(\sim q)$ Law of Double Negation
5. $\sim(\sim q)$ 3, 4, Rule of Substitution
6. $\sim p \rightarrow \sim q$ Given
7. $\sim(\sim p)$ 5, 6, Rule of Modus Tollens
8. $\sim(\sim p) \leftrightarrow p$ Law of Double Negation
9. p 7, 8, Rule of substitution
10. $p \rightarrow t$ Given
11. t 9, 10, Rule of Modus Ponens

Therefore, t.

Example 4

1. $[r \rightarrow (s \rightarrow \sim t)] \wedge [(\sim s \rightarrow \sim r) \wedge r]$ Given
2. $r \rightarrow (s \rightarrow \sim t)$ and $(\sim s \rightarrow \sim r) \wedge r$ 1, Def^n of Conjunction
3. $\sim s \rightarrow \sim r$ and r 2, Def^n of Conjunction
4. $s \rightarrow \sim t$ 2, 3, Rule of Modus Ponens
5. $r \leftrightarrow \sim(\sim r)$ Law of Double Negation
6. $\sim(\sim r)$ 3, 5, Rule of Substitution
7. $\sim(\sim s)$ 3, 6, Rule of Modus Tollens
8. $\sim(\sim s) \leftrightarrow s$ Law of Double Negation
9. s 7, 8, Rule of Substitution
10. $\sim t$ 4, 9, Rule of Modus Ponens

Therefore, $\sim t$.

Example 1: Prove that if $(\sim x \vee \sim y) \rightarrow (z \wedge w)$, $z \rightarrow t$, and $\sim t$, then x.

Example 2: Prove that if $(\sim a \vee b) \rightarrow c$, $\sim c \vee d$, and $d \rightarrow \sim(e \vee \sim e)$, then a

Example 1 We want to prove that x is true so we assume that x is false that is $\sim x$ is true and we want to prove for contradiction

1. $\sim x$ Assumption
2. $\sim x \rightarrow (\sim x \vee \sim y)$ Law of Addition
3. $\sim x \vee \sim y$ 1, 2, Rule of Modus Ponens
4. $(\sim x \vee \sim y) \rightarrow (z \wedge w)$ Given
5. $z \wedge w$ 3, 4, Rule of Modus Ponens
6. z and w 5, Def^{n} of Conjunction
7. $z \rightarrow t$ Given
8. t 6, 7, Rule of modus Ponens
9. $\sim t$ Given
10. $t \wedge \sim t$ 8, 9, Def^{n} of Conjuction

Since $t \wedge \sim t$ is contradiction, hence $\sim x$ is false.

Therefore, x is true.

Example 2 Prove that if $(\sim a \vee b) \to c$, $\sim c \vee d$, and $d \to \sim(e \vee \sim e)$, then a

Sol^n We assume $\sim a$ to prove for the contradiction

1. $\sim a$ Assumption
2. $\sim a \to (\sim a \vee b)$ Law of Addition
3. $\sim a \vee b$ 1,2, Rule of Modus Ponens
4. $(\sim a \vee b) \to c$ Given
5. c 3,4, Rule of Modus Ponens
6. $c \leftrightarrow \sim(\sim c)$ Law of Double Negation
7. $\sim(\sim c)$ 5, 6, Rule of Substitution
8. $\sim c \vee d$ Given
9. d 7, 8, Disjunctive Syllogism
10. $d \to \sim(e \vee \sim e)$ Given
11. $\sim(e \vee \sim e)$ 9, 10, Rule of Modus Ponens
12. $\sim(e \vee \sim e) \leftrightarrow (\sim e \wedge \sim(\sim e))$ De Morgan's Laws
13. $\sim e \wedge \sim(\sim e)$ 11, 12, Rule of Substitution

Since $\sim e \wedge \sim(\sim e)$ is contradiction, hence a is true.

Example 1: Prove that if $x \vee \sim y$, and $z \to \sim(x \vee t)$, then $y \to \sim z$.

Example 2: Prove that if $[a \vee (b \to c)] \wedge (b \vee e)$, then $\sim a \to (\sim c \to e)$.

Example 1

We assume y and prove $\sim z$.

1.	y	Assumption
2.	$y \leftrightarrow \sim(\sim y)$	Law of Double Negation
3.	$\sim(\sim y)$	1, 2, Rule of Substitution
4.	$x \vee \sim y$	Given
5.	x	3, 4, Disjunctive Syllogism
6.	$x \to (x \vee t)$	Law of Addition
7.	$x \vee t$	5, 6, Rule of Modus Ponens
8.	$(x \vee t) \leftrightarrow \sim[\sim(x \vee t)]$	Law of Double Negation
9.	$\sim(\sim(x \vee t))$	7, 8, Rule of Substitution
10.	$z \to \sim(x \vee t)$	Given
11	$\sim z$	9, 10, Rule of Modus Tollens

Hence, $y \to \sim z$

Example 2 Prove that if $[a \vee (b \to c)] \wedge (b \vee e)$, then $\sim a \to (\sim c \to e)$

Proof Assume $\sim a$ to show that $\sim c \to e$. Since $\sim c \to e$ is in the form of " $\to$ ", so we have to assume $\sim c$ to show e.

1. $[a \vee (b \to c] \wedge (b \vee e)$ Given
2. $a \vee (b \to c)$ and $(b \vee e)$ 1, Def^n of Conjunction
3. $\sim a$ Assumption
4. $b \to c$ 2, 3, Disjunctive Syllogism
5. $\sim c$ Assumption
6. $\sim b$ 4, 5, Rule of Modus Tollens
7. e 2, 6, Disjunctive Syllogism

Hence, $\sim a \to (\sim c \to e)$

Example 1: Prove that if $(\sim x \wedge y) \to \sim z$, $\sim(x \vee y) \to w$, and $\sim x$, then $\sim z \vee w$.

Example 2: Prove that if $a \vee (b \wedge \sim c)$, $\sim a$, $b \to (d \to e)$, and $\sim c \to (x \to y)$, then $[a \vee x \vee d] \to (y \vee e)$.

Example 1: In this example, we can either assume $\sim w$ and prove $\sim z$ or assume $\sim(\sim z)$ and prove w. In this case we will assume $\sim(\sim z)$ to show w.

Proof

1. $\sim(\sim z)$ — Assumption
2. $(\sim x \wedge y) \to \sim z$ — Given
3. $\sim(\sim x \wedge y)$ — 1, 2, Rule of Modus Tollens
4. $\sim(\sim x \wedge y) \leftrightarrow [\sim(\sim x) \vee \sim y]$ — De Morgan's Laws
5. $\sim(\sim x) \vee \sim y$ — 3, 4, Rule of Substitution
6. $[\sim(\sim x) \vee \sim y] \leftrightarrow [x \vee \sim y]$ — Law of Double Negation
7. $x \vee \sim y$ — 5, 6, Rule of Substitution
8. $\sim x$ — Given
9. $\sim y$ — 7, 8, Disjunctive Syllogism
10. $\sim x \wedge \sim y$ — 8, 9, Def^{n} of Conjunction
11. $(\sim x \wedge \sim y) \leftrightarrow \sim(x \vee y)$ — De Morgan's Laws
12. $\sim(x \vee y)$ — 10, 11, Rule of Substitution
13. $\sim(x \vee y) \to w$ — Given
14. w — 12, 13, Rule of Modus Ponens

Example 2 Prove that if $a \vee (b \wedge \sim c)$, $\sim a$, $b \to (d \to e)$, and $\sim c \to (x \to y)$, then $[a \vee x \vee d] \to (y \vee e)$

Proof We will assume $a \vee x \vee d$ and prove $y \vee e$.

To prove $y \vee e$, we assume $\sim y$ and show e

1.	$a \vee (b \wedge \sim c)$	Given
2.	$\sim a$	Given
3.	$b \wedge \sim c$	1, 2, Disjunctive Syllogism
4.	b and $\sim c$	3, Def^n of Conjunction
5.	$b \to (d \to e)$	Given
6.	$d \to e$	4, 5, Rule of Modus Ponens
7.	$a \vee x \vee d$	Assumption
8.	$x \vee d$	2, 7, Disjunctive Syllogism
9.	$\sim c \to (x \to y)$	Given
10.	$x \to y$	4, 9, Rule of Modus Ponens
11	$\sim y$	Assumption
12.	$\sim x$	10, 11, Rule of Modus Tollens
13.	d	8, 12, Disjunctive Syllogism
14.	e	6, 13, Rule of Modus Ponens

Hence, $y \vee e$

Example : Prove that if $a \to (b \wedge c)$, $(d \wedge c) \to (e \wedge \sim a)$, $a \vee (d \to \sim e)$, and $d \wedge (b \to e)$, then $\sim e \leftrightarrow \sim a$

Proof In order to prove $\sim e \leftrightarrow \sim a$, we need to prove $\sim e \to \sim a$ and $\sim a \to \sim e$.

First, we prove $\sim e \to \sim a$ by assuming $\sim e$ and show $\sim a$.

1.	$\sim e$	Assumption
2.	$\sim e \to (\sim e \vee a)$	Law of Addition
3.	$\sim e \vee a$	1, 2, Rule of Modus Ponens
4.	$(\sim e \vee a) \leftrightarrow \sim(e \wedge \sim a)$	De Morgan's Law
5.	$\sim(e \wedge \sim a)$	3, 4, Rule of Substitution
6.	$(d \wedge c) \to (e \wedge \sim a)$	Given
7.	$\sim(d \wedge c)$	5, 6, Rule of Modus Tollens
8.	$\sim(d \wedge c) \leftrightarrow [\sim d \vee \sim c]$	De Morgan's Laws
9.	$\sim d \vee \sim c$	7, 8, Rule of Substitution
10.	$d \wedge (b \to e)$	Given
11.	d and $(b \to e)$	10, Def^{n} of Conjunction
12.	$d \leftrightarrow \sim(\sim d)$	Law of Double Negation
13.	$\sim(\sim d)$	11, 12, Rule of Substitution
14.	$\sim c$	9, 13, Disjunctive Syllogism
15.	$\sim c \to (\sim c \vee \sim b)$	Law of Addition
16	$\sim c \vee \sim b$	14, 15, Rule of Modus Ponens
17.	$(\sim c \vee \sim b) \leftrightarrow \sim(c \wedge b)$	De Morgan's Laws
18.	$\sim(c \wedge b) \leftrightarrow \sim(b \wedge c)$	Commutative Laws
19.	$\sim(b \wedge c)$	16, 17, 18, Rule of Substitution
20.	$a \to (b \wedge c)$	Given
21.	$\sim a$	19, 20, Rule of Modus Tollens

Hence, $\sim e \to \sim a$

(Continued →)

Next, we prove $\sim a \rightarrow \sim e$ by assuming $\sim a$ and show $\sim e$

1. $\sim a$ Assumption
2. $a \vee (d \rightarrow \sim e)$ Given
3. $d \rightarrow \sim e$ 1, 2, Disjunctive Syllogism
4. $d \wedge (b \rightarrow e)$ Given
5. d and $b \rightarrow e$ 4, Def^{n} of Conjunction
6. $\sim e$ 3, 5, Rule of Modus Ponens

Hence, $\sim a \rightarrow \sim e$

$\therefore$ We conclude that $\sim e \leftrightarrow \sim a$

1. Prove that if $[\sim(s\wedge r)\vee s\vee p]\to(s\wedge t)$, then t.

Proof

1) $s \vee \sim s$ — Law of Excluded Middle

2) $(s \vee \sim s) \to [(s \vee \sim s) \vee (p \vee \sim r)]$ — Law of Addition

3) $(s \vee \sim s) \vee (p \vee \sim r)$ — 1, 2, Rule of Modus Ponens

4) $[(s \vee \sim s) \vee (p \vee \sim r)] \leftrightarrow [(\sim s \vee \sim r) \vee s \vee p]$ — Associative Laws and Commutative Law

5) $(\sim s \vee \sim r) \vee s \vee p$ — 4, 5, Rule of Substitution

6) $[(\sim s \vee \sim r) \vee s \vee p] \leftrightarrow [\sim(s \wedge r) \vee s \vee p]$ — De Morgan's Laws

7) $\sim(s \wedge r) \vee s \vee p$ — 5, 6, Rule of Substitution

8) $[\sim(s \wedge r) \vee s \vee p] \to (s \wedge t)$ — Given

9) $s \wedge t$ — 7, 8, Rule of Modus Ponens

10) t — 9, Def^{n} of Conjunction

Hence, t.

2. Prove that if $a \vee b \vee c$, $\sim(d \wedge e) \rightarrow \sim(f \vee a)$, and $(c \rightarrow a) \wedge (\sim b \vee f)$, then d.

Proof Assume $\sim d$

1)	$\sim d$	Assumption
2)	$\sim d \rightarrow (\sim d \vee \sim e)$	Law of Addition
3)	$\sim d \vee \sim e$	1, 2, Rule of Modus Ponens
4)	$(\sim d \vee \sim e) \leftrightarrow \sim(d \wedge e)$	De Morgan's Laws
5)	$\sim(d \wedge e)$	3, 4, Rule of Substitution
6)	$\sim(d \wedge e) \rightarrow \sim(f \vee a)$	Given
7)	$\sim(f \vee a)$	5, 6, Rule of Modus Ponens
8)	$\sim(f \vee a) \leftrightarrow (\sim f \wedge \sim a)$	De Morgan's Laws
9)	$\sim f \wedge \sim a$	7, 8, Rule of Substitution
10)	$\sim f$ and $\sim a$	9, Def^{n} of Conjunction
11)	$(c \rightarrow a) \wedge (\sim b \vee f)$	Given
12)	$(c \rightarrow a)$ and $(\sim b \vee f)$	11, Def^{n} of Conjunction
13)	$\sim c$	10, 12, Rule of Modus Tollens
14)	$\sim b$	10, 12, Disjunctive Syllogism
15)	$a \vee b \vee c$	Given
16)	a	13, 14, 15, Disjunctive Syllogism
17)	$a \wedge \sim a$	10, 16, Def^{n} of Conjunction

Since $a \wedge \sim a$ is contradiction, hence we conclude d.

3. Prove that if $\sim(p \to q)$, $(p \wedge r) \to (s \to q)$, and $(\sim r \to q) \vee (p \to s)$, then $r \leftrightarrow \sim s$.

Proof Assume r to show $\sim s$

1)	$\sim(p \to q)$	Given
2)	$\sim(p \to q) \leftrightarrow (p \wedge \sim q)$	Law of Negation for Implication
3)	$p \wedge \sim q$	1, 2, Rule of Substitution
4)	p and $\sim q$	3, Def^{n} of Conjunction
5)	r	Assumption
6)	$p \wedge r$	4, 5, Def^{n} of Conjunction
7)	$(p \wedge r) \to (s \to q)$	Given
8)	$s \to q$	6, 7, Rule of Modus Ponens
9)	$\sim s$	4, 8, Rule of Modus Tollens

Hence, $r \to \sim s$.

Assume $\sim s$ to show r.

1)	$\sim(p \to q)$	Given
2)	$\sim(p \to q) \leftrightarrow (p \wedge \sim q)$	Law of Negation for Implication
3)	$p \wedge \sim q$	1, 2, Rule of Substitution
4)	p and $\sim q$	3, Def^{n} of Conjunction
5)	$\sim s$	Assumption
6)	$p \wedge \sim s$	4, 5, Def^{n} of Conjunction
7)	$(p \wedge \sim s) \leftrightarrow \sim(p \to s)$	Law of Negation for Implication
8)	$\sim(p \to s)$	6, 7, Rule of Substitution
9)	$(\sim r \to q) \vee (p \to s)$	Given
10)	$\sim r \to q$	8, 9, Disjunctive Syllogism
11)	$\sim(\sim r)$	4, 10, Rule of Modus Tollens
12)	$\sim(\sim r) \leftrightarrow r$	Law of Double Negation
13)	r	11, 12, Rule of Substitution

Example : Show that for every natural number n, $n<2^n$.

<u>Proof</u> We proceed by induction. Let S_n be the statement " $n<2^n$ where n is natural number."

1.) S_1 is " $1<2^1$ " which is true.

2) Assume that S_k is true, that is, we assume $k<2^k$ where k is a natural number is true. We need to show that S_{k+1} is true. Observe that

$$2^{k+1} = 2\cdot 2^k > 2k = k+k \geqslant k+1$$

Hence, S_{k+1} is true.

$\therefore$ For every natural number n, $n<2^n$.

Example : Let n be positive integer that is greater than 4, prove that $n^2+1<2^n$.

<u>Proof</u> Let S_n be the statement " $n^2+1<2^n$ where $n \in I^+$ and $n \geqslant 5$ "

1) S_5 is " $5^2+1 = 26 < 32 = 2^5$ " which is true.

2) Assume S_k is true, that is " $k^2+1<2^k$ where $k\in\mathbb{N}$ and $k\geqslant 5$ "

We need to show that S_{k+1} is true, that is to show $(k+1)^2+1<2^{k+1}$

Since $k^2+1<2^k$, we have

$$2(k^2+1) < 2\cdot 2^k$$
$$2k^2+2 < 2^{k+1}$$
$$k^2+k^2+2 < 2^{k+1}$$
$$k^2+k\cdot k+2 < 2^{k+1}$$
$$k^2+2k+2 < 2^{k+1} \quad \text{b/c } 2<5\leq k$$
$$(k^2+2k+1)+1 < 2^{k+1}$$
$$(k+1)^2+1 < 2^{k+1}$$

$\therefore$ S_{k+1} is true. Hence, by induction, we conclude that $n^2+1<2^n$ where n is any positive integer that is greater than 4.