Macroeconomics

Lecture 7.2

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The Term Structure of Interest Rate

- What is the relationship between the level of an equilibrium risk free interest rate and its term to maturity?

The Term Structure of Interest Rate

- Suppose that there are markets in one-period and two-period perfectly safe loans.
- R_{1t} and R_{2t} are risk free gross rate of return from the one- and two-period safe loans, respectively. R_{1t} and R_{2t} are known with certainty at time t.
- $\therefore R_{1t}^{-1}$ is the price of a perfectly sure claim to 1 unit of consumption at time t+1.

$$Max \qquad E_0 \sum_{t=0}^{\infty} \beta^t U(c_t),$$

$$s.t. \qquad c_t + L_{1t} + L_{2t} \le d_t + L_{1t-1} R_{1t-1} + L_{2t-2} R_{2t-2},$$

$$where \qquad L_{jt} \ is \ the \ amount \ lent \ for \ j \ periods \ at \ time \ t.$$

$$The \ Lagrangian \ function \ is,$$

$$J = E_0 \sum_{t=0}^{\infty} \beta^t \left[U(c_t) + \lambda_t \left(d_t + L_{1t-1} R_{1t-1} + L_{2t-2} R_{2t-2} - c_t - L_{1t} - L_{2t} \right) \right]$$

$$The \ first-order \ necessary \ conditions \ are$$

$$\frac{\partial J}{\partial c_t} = 0, \Rightarrow \qquad \left[\partial U(c_t) / \partial c_t \right] - \lambda_t = 0,$$

$$\frac{\partial J}{\partial L_{1t}} = 0, \Rightarrow \qquad -\lambda_t + \beta E_t \lambda_{t+1} R_{1t} = 0,$$

$$\frac{\partial J}{\partial L_{2t}} = 0, \Rightarrow \qquad -\lambda_t + \beta^2 E_t \lambda_{t+2} R_{2t} = 0,$$

Substituting the first into each of the second and third conditions gives

$$E_t\left[\beta\frac{\left(\partial U\left(c_{t+1}\right)/\partial c_{t+1}\right)}{\left(\partial U\left(c_t\right)/\partial c_t\right)}R_{1t}\right]=1, \qquad (1)$$

$$E_t\left[\beta^2\frac{\left(\partial U\left(c_{t+2}\right)/\partial c_{t+2}\right)}{\left(\partial U\left(c_t\right)/\partial c_t\right)}R_{2t}\right]=1. \qquad (2)$$

Since R_{1t} *and* R_{2t} *are known with certainty at the beginning of* t,

$$\therefore R_{1t}^{-1}=E_t\beta\frac{\left(\partial U\left(c_{t+1}\right)/\partial c_{t+1}\right)}{\left(\partial U\left(c_t\right)/\partial c_t\right)} = \textit{the price of a perfectly sure claim to}$$

1 *unit of consumption at time* $t+1$.

This price can be easily obtained from the following reasoning,

Since R_{1t} *units of consumption at* $t+1$ = 1 *unit of consumption at time* t

hence, 1 " " " = R_{1t}^{-1} " " "

$$R_{2t}^{-1}=E_t\beta^2\frac{\left(\partial U\left(c_{t+2}\right)/\partial c_{t+2}\right)}{\left(\partial U\left(c_t\right)/\partial c_t\right)} = \textit{the price of a perfectly sure claim to}$$

1 *unit of consumption at time* $t+2$.

$$Let\ us\ write\ Eq\,(1)\ as$$

$$\beta\frac{\left(\partial U\left(c_{t+1}\right)/\partial c_{t+1}\right)}{\left(\partial U\left(c_{t}\right)/\partial c_{t}\right)}R_{1t}=1+\varepsilon_{1t+1},$$

$$where\ \ E_t\varepsilon_{1t+1}\ \ is\ the\ least\ squares\ residual\ \ and\ \ E_t\varepsilon_{1t+1}=0.$$

$$\therefore\ \beta\frac{\left(\partial U\left(c_{t+1}\right)/\partial c_{t+1}\right)}{\left(\partial U\left(c_{t}\right)/\partial c_{t}\right)}=R_{1t}^{-1}+R_{1t}^{-1}\varepsilon_{1t+1}\qquad(3)$$

$$Eq\ (2)\ can\ be\ written\ as$$

$$R_{2t}^{-1}=E_t\left[\beta^2\frac{\left(\partial U\left(c_{t+1}\right)/\partial c_{t+1}\right)}{\left(\partial U\left(c_{t}\right)/\partial c_{t}\right)}\frac{\left(\partial U\left(c_{t+2}\right)/\partial c_{t+2}\right)}{\left(\partial U\left(c_{t+1}\right)/\partial c_{t+1}\right)}\right]\qquad(4)$$

$$Substituting\ \ Eq\ (3)\ \mathrm{int}\,o\ \ Eq\ (4)\ gives$$

$$R_{2t}^{-1}=E_t\left[\left(R_{1t}^{-1}+R_{1t}^{-1}\varepsilon_{1t+1}\right)\left(R_{1t+1}^{-1}+R_{1t+1}^{-1}\varepsilon_{1t+2}\right)\right]$$

$$=R_{1t}^{-1}E_t\left[R_{1t+1}^{-1}+R_{1t+1}^{-1}\varepsilon_{1t+1}+R_{1t+1}^{-1}\varepsilon_{1t+1}\varepsilon_{1t+2}+R_{1t+1}^{-1}\varepsilon_{1t+2}\right].$$

$$Or,\qquad R_{2t}^{-1}=R_{1t}^{-1}\left[E_tR_{1t+1}^{-1}+E_tR_{1,t+1}^{-1}E\varepsilon_{t_{1t+1}}+\mathrm{cov}_t\left(R_{1t+1}^{-1},\varepsilon_{1t+1}\right)\right]$$

$$+R_{1t}^{-1}E_tR_{1t+1}^{-1}\varepsilon_{1t+1}\varepsilon_{1t+2}+R_{1t}^{-1}E_tR_{1t+1}^{-1}\varepsilon_{1t+2}\qquad(5)$$

Consider the last two terms

$$E_t R_{1t+1}^{-1}\varepsilon_{1t+1}\varepsilon_{1t+2} = E_t E_{t+1} R_{1t+1}^{-1}\varepsilon_{1t+1}\varepsilon_{1t+2}$$
$$= E_t R_{1t+1}^{-1} E_{t+1}\varepsilon_{1t+1}\varepsilon_{1t+2}$$
$$= E_t R_{1t+1}^{-1} E_{t+1}\varepsilon_{1t+1} E_{t+1}\varepsilon_{1t+2} = 0,$$

$$E_t R_{1t+1}^{-1}\varepsilon_{1t+2} = E_t E_{t+1} R_{1t+1}^{-1}\varepsilon_{1t+2} = E_t R_{1t+1}^{-1} E_{t+1}\varepsilon_{1t+2} = 0.$$

Substituting these 2 equalities into (5) gives

$$R_{2t}^{-1} = R_{1t}^{-1}\left[E_t R_{1t+1}^{-1} + \operatorname{cov}_t\left(R_{1t+1}^{-1}, \varepsilon_{1t+1}\right)\right], \qquad (6)$$

where $\varepsilon_{1t+1} = \beta \dfrac{\left(\partial U\left(c_{t+1}\right)/\partial c_{t+1}\right)}{\left(\partial U\left(c_t\right)/\partial c_t\right)} R_{1t} - 1.$

For example,

$$\uparrow R_{1t+1}^{-1} \Rightarrow \downarrow c_{t+2}, \uparrow c_{t+1} \Rightarrow \downarrow \frac{\partial U\left(c_{t+1}\right)}{\partial c_{t+1}} \text{ (since utility is concave)} \Rightarrow \downarrow \varepsilon_{1t+1},$$

hence, $\operatorname{cov}_t\left(R_{1t+1}^{-1}, \varepsilon_{1t+1}\right) < 0,$ *so from* (6),

$$R_{1t} = R_{2t}.\left[E_t R_{1t+1}^{-1} + \operatorname{cov}_t\left(R_{1t+1}^{-1}, \varepsilon_{1t+1}\right)\right],$$

$$R_{1t} E_t R_{1t+1} = R_{2t} + \operatorname{cov}_t\left(R_{1t+1}^{-1}, \varepsilon_{1t+1}\right) R_{2t} E_t R_{1t+1} < R_{2t}$$

Eq.(6) *is a generalized version of the pure expectation theory of the term structure of interest rates, adjusted for the risk premium* $\operatorname{cov}_t\left(R_{1t+1}^{-1}, \varepsilon_{1t+1}\right)$.

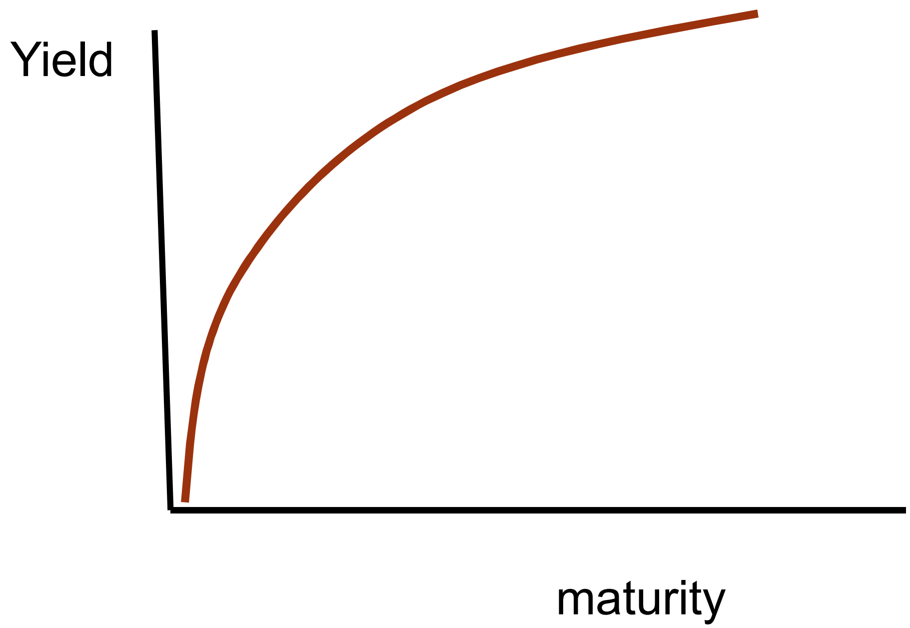
If the private agent is risk aversive, then $\operatorname{cov}_t\left(R_{1t+1}^{-1}, \varepsilon_{1t+1}\right) < 0$.

If the private agent is risk lover, then $\operatorname{cov}_t\left(R_{1t+1}^{-1}, \varepsilon_{1t+1}\right) > 0$.

The pure expectation theory states that $R_{2t}^{-1} = R_{1t}^{-1}\left[E_t R_{1t+1}^{-1}\right]$, *this is the result from the assumption that private agent is risk neutral.*

Homework

- What is Quantitative Easing (QE)?
- How could the yield curve be flatten by QE?



Contingent Claim Market

Pricing of one-step-ahead state-contingent securities.

- Let the state of the economy evolve according to a Markov process described by density f(x',x), hence,

$$prob(x_{t+1} \le x' \mid x_t = x) = \int_{-\infty}^{x'} f(u,x)\,du \equiv F(x',x)$$

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Contingent Claim Market

- Let Ω be the entire space of possible x.
- A is a subset in Ω

Contingent Claim Market

- In period t, given that the economy is in state x_t, then one can purchase, or sell, a claim to 1 unit of next period consumption good contingent on the event that x_{t+1} belongs to a set A, at the following price (measured in terms of time t consumption)

$$\int_{x_{t+1} \in A} q\left(x_{t+1}, x_t\right) dx_{t+1}$$

Contingent Claim Market

- **Let Ω be the entire space of possible x, the time t, price of a perfectly certain claim on period (t+1) consumption is**

$$\int_{x_{t+1}\in\Omega} q\left(x_{t+1},x_t\right)dx_{t+1} = \frac{1}{R_{1t}}$$

$$= \textit{the price of a perfectly sure claim to } 1 \textit{ unit of consumption at time } t+1,$$

where

$$\int_{x_{t+1}\in\Omega} q\left(x_{t+1},x_t\right)dx_{t+1} > \int_{x_{t+1}\in A} q\left(x_{t+1},x_t\right)dx_{t+1}.$$