Chapter 4 Determinants

4.1 Motivations for the Definition of Determinants

Consider a system of two linear equations with two variables

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

Without loss of generality, we can assume that the coefficient $a_{11} \neq 0$ and perform an elementary row operation to get

$$\begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} - \dfrac{a_{21}}{a_{11}} a_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - \dfrac{a_{21}}{a_{11}} b_1 \end{bmatrix},$$

and we solve for and as,

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}},$$
$$x_1 = \frac{a_{22}b_1 - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}.$$

We can uniquely compute the values of and if, and only if,

$$a_{11}a_{22} - a_{12}a_{21} \neq 0. \qquad (4.1)$$

With a system of three equations and three variables, the condition becomes,

$$a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \neq 0. \qquad (4.2)$$

See **Simon & Blume** [1994] pages 720-721.

The left-hand sides of (4.1) and (4.2) are called the determinants of 2×2 and 3×3 matrices respectively. We will extract patterns from (4.1) and (4.2) to arrive at the general definition for $n \times n$ matrices. Later it will be shown how the determinant is used to solve systems of linear equations.

Pattern 1 Terms in (4.1) and (4.2) contains exactly one element from each row and each column of the matrices. Each contains all possible such terms and we have 2! terms in (4.1) and 3! in (4.2).

Pattern 2 The sign of each term depends on the permutation of the subscripts of elements in that term. Specifically, if we order the elements by their first subscript as in (4.1) and (4.2), the minimum number of interchanges of the adjacent second subscripts until they are in ascending order determines the sign of that term. If this number is odd, then the sign is negative, and positive otherwise. A permutation is called either odd or even according to this number.

For example, the term $a_{11}a_{22}$ has the second subscripts in the order $\{1,2\}$. No interchange is needed and the sign is thus positive. The term $a_{12}a_{21}$ has the second subscripts in the order $\{2,1\}$, thus one interchange is needed and the sign is negative. From (4.2), the fourth term, for example, is $a_{12}a_{23}a_{31}$ and the second subscripts are in the order $\{2,3,1\}$. Its sign is positive since we need two interchanges.

> ***Problem** For a given permutation, is there a minimum number of interchanges until the ascending order is attained? Prove that there is such a minimum by induction. Then, we take the number of interchanges discussed above to mean this minimum. See **Fraleigh & Beauregard** [1995], page 263, #37.

Pattern 3 The left-hand side of (4.2) can be simplified as,

$$a_{11}\left(a_{22}a_{33}-a_{23}a_{32}\right)-a_{12}\left(a_{21}a_{33}-a_{23}a_{31}\right)+a_{13}\left(a_{21}a_{32}-a_{22}a_{31}\right).$$

Observe that the terms in brackets are just determinants of matrices,

$$\begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}, \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}, \text{ and } \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix},$$

which are *submatrices* of the original matrix. A formal definition of submatrix is given as,

Definition 4.1 Let **A** be a square matrix in $\boldsymbol{R}^{n\times n}$. The ***submatrix*** (i,j) of **A**, denoted by , $\mathbf{A}_{ij}$ is just matrix **A** but with row i and column j deleted. So submatrices (i,j) of **A** have $n-1$ rows and $n-1$ columns.

The determinant of $\mathbf{A}\in\boldsymbol{R}^{3\times 3}$ above is just the sum of elements multiplied by the determinants of the submatrices $(1,j)$, $j=1,2,3$. The sign of each term is determined by $(-1)^{1+j}$.

With these patterns we can now define the determinant for any square matrix.

Definition 4.2 Let **A** be a matrix in $\boldsymbol{R}^{n\times n}$. The ***determinant*** of **A**, denoted by $|\mathbf{A}|$ or det **A**, is given by,

$$
\begin{aligned}
|\mathbf{A}| &= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \\
&= \sum_{j=1}^{n} a_{1j}(-1)^{1+j} M_{1j} \\
&= \sum_{j=1}^{n} a_{1j}\Delta_{1j},
\end{aligned}
$$

where is the $(i,j)^{\text{th}}$ ***minor*** of **A** which is the determinant of the submatrix $\mathbf{A}_{ij}$. The term $\Delta_{ij} = (-1)^{i+j} M_{ij}$ is called the $(i,j)^{\text{th}}$ ***cofactor*** of **A**.

Note that if $\mathbf{A} = [a]_{1\times 1}$, then $|\mathbf{A}| = a$.

Example The determinant of an identity matrix is one. The determinant of a diagonal or triangular matrix is the product of its diagonal elements. The determinant of a matrix is as given by the third pattern above. The determinant of a 4×4 matrix is given by

$$
\begin{aligned}
|\mathbf{A}| &= \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \\
&= \sum_{j=1}^{4} a_{1j}(-1)^{1+j} M_{1j} \\
&= a_{11}(-1)^{1+1} M_{11} + a_{12}(-1)^{1+2} M_{12} + \\
&\quad a_{13}(-1)^{1+3} M_{13} + a_{14}(-1)^{1+4} M_{14} \\
&= a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + \\
&\quad a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{33} & a_{44} \end{vmatrix} - a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{33} & a_{43} \end{vmatrix}.
\end{aligned}
$$

This definition expands the determinant along the first row of **A** as in the third pattern. The following theorem establishes that the determinant of **A** can be computed by expanding along any row or column.

Theorem 4.1 Let **A** be a matrix in $\boldsymbol{R}^{n\times n}$. Then,

$$|\mathbf{A}| = \begin{cases} \sum_{j=1}^{n} a_{ij}\Delta_{ij}, & i = 1,2,\ldots,n, \\ \sum_{i=1}^{n} a_{ij}\Delta_{ij}, & j = 1,2,\ldots,n. \end{cases}$$

Proof See **Simon & Blume** [1994] , Appendix of Chapter 26, pages 743-746.❑

The first equation represents determinant computation by row i expansion, and the second by the column j expansion.

Example The determinant of $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ as expanded along the second column is given by,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$
$$= -a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{22}(a_{11}a_{33} - a_{31}a_{13}) - a_{32}(a_{11}a_{23} - a_{21}a_{13})$$
$$= -a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{12}a_{11}a_{33} - a_{12}a_{31}a_{13} - a_{32}a_{11}a_{23} - a_{32}a_{21}a_{13}.$$

By rearranging terms, we will have the determinant as given in (4.2).

4.2 Properties of Determinants

Theorem 4.2 Let **A** be a matrix in $\boldsymbol{R}^{n\times n}$ and $\mathbf{a}_{\cdot j}$ be its j^{th} column, so that

$$\mathbf{A} = [\mathbf{a}_{\cdot 1} \quad \cdots \quad \mathbf{a}_{\cdot j} \quad \cdots \quad \mathbf{a}_{\cdot n}].$$

Then,

a) If $\mathbf{a}_{\cdot j} = \mathbf{0}$, $j = 1,2,\ldots,n$, then $|\mathbf{A}| = 0$.

b) If $\mathbf{a}_{\cdot j} = \mathbf{a}'_{\cdot j} + \mathbf{a}''_{\cdot j}$, $j = 1,2,\ldots,n$, then,

$$|\mathbf{a}_{\cdot 1} \quad \cdots \quad \mathbf{a}_{\cdot j} \quad \cdots \quad \mathbf{a}_{\cdot n}| = |\mathbf{a}_{\cdot 1} \quad \cdots \quad \mathbf{a}'_{\cdot j} \quad \cdots \quad \mathbf{a}_{\cdot n}| + |\mathbf{a}_{\cdot 1} \quad \cdots \quad \mathbf{a}''_{\cdot j} \quad \cdots \quad \mathbf{a}_{\cdot n}|.$$

c) For any scalar c and $j = 1,2,\ldots,n$, $\begin{vmatrix}\mathbf{a}_{\cdot 1} & \cdots & c\mathbf{a}_{\cdot j} & \cdots & \mathbf{a}_{\cdot n}\end{vmatrix} = c|\mathbf{A}|$.

d) If $\mathbf{a}_{\cdot j} = \mathbf{a}_{\cdot k}$ for some $j \neq k$, then $|\mathbf{A}| = 0$.

e) For any pair of columns j and k, $j \neq k$,

$$\begin{vmatrix}\mathbf{a}_{\cdot 1} & \cdots & \mathbf{a}_{\cdot j} + c\mathbf{a}_{\cdot k} & \cdots & \mathbf{a}_{\cdot k} & \cdots & \mathbf{a}_{\cdot n}\end{vmatrix} = \begin{vmatrix}\mathbf{a}_{\cdot 1} & \cdots & \mathbf{a}_{\cdot j} & \cdots & \mathbf{a}_{\cdot k} & \cdots & \mathbf{a}_{\cdot n}\end{vmatrix} = |\mathbf{A}|.$$

f) For any pair of columns j and k, $j \neq k$,

$$|\mathbf{A}| = \begin{vmatrix}\mathbf{a}_{\cdot 1} & \cdots & \mathbf{a}_{\cdot j} & \cdots & \mathbf{a}_{\cdot k} & \cdots & \mathbf{a}_{\cdot n}\end{vmatrix} = -\begin{vmatrix}\mathbf{a}_{\cdot 1} & \cdots & \mathbf{a}_{\cdot k} & \cdots & \mathbf{a}_{\cdot j} & \cdots & \mathbf{a}_{\cdot n}\end{vmatrix}.$$

g) $|\mathbf{A}^{\mathrm{T}}| = |\mathbf{A}|$.

Proof

a) Expanding along the j^{th} column, we have $|\mathbf{A}| = \sum_{i=1}^{n} 0\Delta_{ij} = 0$.

b) Expanding along the j^{th} column, we have

$$\begin{aligned}
\begin{vmatrix}\mathbf{a}_{\cdot 1} & \cdots & \mathbf{a}'_{\cdot j} + \mathbf{a}''_{\cdot j} & \cdots & \mathbf{a}_{\cdot n}\end{vmatrix} &= \sum_{i=1}^{n}\left(a'_{ij} + a''_{ij}\right)\Delta_{ij} \\
&= \sum_{i=1}^{n} a'_{ij}\Delta_{ij} + \sum_{i=1}^{n} a''_{ij}\Delta_{ij} \\
&= \begin{vmatrix}\mathbf{a}_{\cdot 1} & \cdots & \mathbf{a}'_{\cdot j} & \cdots & \mathbf{a}_{\cdot n}\end{vmatrix} + \begin{vmatrix}\mathbf{a}_{\cdot 1} & \cdots & \mathbf{a}''_{\cdot j} & \cdots & \mathbf{a}_{\cdot n}\end{vmatrix}.
\end{aligned}$$

c) Expanding along the j^{th} column, we have

$$\begin{vmatrix}\mathbf{a}_{\cdot 1} & \cdots & c\mathbf{a}_{\cdot j} & \cdots & \mathbf{a}_{\cdot n}\end{vmatrix} = \sum_{i=1}^{n} ca_{ij}\Delta_{ij} = c\sum_{i=1}^{n} a_{ij}\Delta_{ij} = c|\mathbf{A}|.$$

d) We will prove by induction on the size of the matrix **A**. For matrix **A** of order 2×2, this is obviously true. Assume that it also true for any matrix **A** of order $n \times n$, then we will show for the case of any $(n+1) \times (n+1)$.

Thus let **A** now be a matrix in $\boldsymbol{R}^{(n+1)\times(n+1)}$ with columns j and k, $j \neq k$, being identical. Compute the determinant of **A** by expanding along any column of **A** that is not j or k. This is always possible as we are now considering matrices with more than two columns. We can assume without loss of generality that $j \neq k \neq 1$, and expand along the first column. So,

$$\left|\mathbf{A}\right| = \sum_{p=1}^{n+1} a_{p1}\left(-1\right)^{p+1} M_{p1}.$$

The minor M_{p1} is just the determinant of the submatrix $\mathbf{A}_{p1}$, which is the matrix **A** with the first column and row *p* deleted. This submatrix is of order $n \times n$, and more importantly it has columns and being identical. By the induction hypothesis, $M_{p1} = 0$, for $p = 1,2,\ldots,n$, and then $\left|\mathbf{A}\right| = 0$.

e) By **(b)**, we have

$$\begin{aligned}
\left|\mathbf{a}_{\cdot 1} \ \cdots \ \mathbf{a}_{\cdot j} + c\mathbf{a}_{\cdot k} \ \cdots \ \mathbf{a}_{\cdot k} \ \cdots \ \mathbf{a}_{\cdot n}\right| &= \left|\mathbf{a}_{\cdot 1} \ \cdots \ \mathbf{a}_{\cdot j} \ \cdots \ \mathbf{a}_{\cdot k} \ \cdots \ \mathbf{a}_{\cdot n}\right| + \\
&\quad \left|\mathbf{a}_{\cdot 1} \ \cdots \ c\mathbf{a}_{\cdot k} \ \cdots \ \mathbf{a}_{\cdot k} \ \cdots \ \mathbf{a}_{\cdot n}\right| \\
&= \left|\mathbf{a}_{\cdot 1} \ \cdots \ \mathbf{a}_{\cdot j} \ \cdots \ \mathbf{a}_{\cdot k} \ \cdots \ \mathbf{a}_{\cdot n}\right| \\
&= \left|\mathbf{A}\right|,
\end{aligned}$$

where $\left|\mathbf{a}_{\cdot 1} \ \cdots \ c\mathbf{a}_{\cdot k} \ \cdots \ \mathbf{a}_{\cdot k} \ \cdots \ \mathbf{a}_{\cdot n}\right| = 0$ by **(c)** and **(d)**.

f) We can show by induction argument similar to the one in **(d)**, or we can use **(b)**, **(c)**, **(d)** and **(e)** as follows.

$$\begin{aligned}
\left|\mathbf{A}\right| &= \left|\mathbf{a}_{\cdot 1} \ \cdots \ \mathbf{a}_{\cdot j} \ \cdots \ \mathbf{a}_{\cdot k} \ \cdots \ \mathbf{a}_{\cdot n}\right| \\
&= \left|\mathbf{a}_{\cdot 1} \ \cdots \ \mathbf{a}_{\cdot j} + \mathbf{a}_{\cdot k} \ \cdots \ \mathbf{a}_{\cdot k} \ \cdots \ \mathbf{a}_{\cdot n}\right| \\
&= \left|\mathbf{a}_{\cdot 1} \ \cdots \ \mathbf{a}_{\cdot j} + \mathbf{a}_{\cdot k} \ \cdots \ \mathbf{a}_{\cdot k} - \left(\mathbf{a}_{\cdot j} + \mathbf{a}_{\cdot k}\right) \ \cdots \ \mathbf{a}_{\cdot n}\right| \\
&= \left|\mathbf{a}_{\cdot 1} \ \cdots \ \mathbf{a}_{\cdot j} + \mathbf{a}_{\cdot k} \ \cdots \ -\mathbf{a}_{\cdot j} \ \cdots \ \mathbf{a}_{\cdot n}\right| \\
&= \left|\mathbf{a}_{\cdot 1} \ \cdots \ \mathbf{a}_{\cdot j} + \mathbf{a}_{\cdot k} - \mathbf{a}_{\cdot j} \ \cdots \ -\mathbf{a}_{\cdot j} \ \cdots \ \mathbf{a}_{\cdot n}\right| \\
&= \left|\mathbf{a}_{\cdot 1} \ \cdots \ \mathbf{a}_{\cdot k} \ \cdots \ -\mathbf{a}_{\cdot j} \ \cdots \ \mathbf{a}_{\cdot n}\right| \\
&= -\left|\mathbf{a}_{\cdot 1} \ \cdots \ \mathbf{a}_{\cdot k} \ \cdots \ \mathbf{a}_{\cdot j} \ \cdots \ \mathbf{a}_{\cdot n}\right|.
\end{aligned}$$

The second, the third and the fifth equations are true by **(e)**, the last is true by **(c)**, and the other equalities are just simplifications.

g) It is trivially true if the dimension of **A** is 2×2. Use the induction hypothesis that it is true for any matrix of dimension $n \times n$ or less, and show that **(g)** holds for any matrix of dimension $\left(n+1\right) \times \left(n+1\right)$. Let **A** be a matrix in $\boldsymbol{R}^{\left(n+1\right)\times\left(n+1\right)}$, and compute $\left|\mathbf{A}^{\mathrm{T}}\right|$ by expanding along its first column,

$$\left|\mathbf{A}^{\mathrm{T}}\right| = \sum_{i=1}^{n+1} a_{i1}^{\mathrm{T}} (-1)^{i+1} M_{i1}^{\mathrm{T}} ,$$

where M_{i1}^{T} is the $(i,1)^{\text{th}}$ minor of $\mathbf{A}^{\mathrm{T}}$, which is the determinant of the $(i,1)^{\text{th}}$ submatrix of $\mathbf{A}^{\mathrm{T}}$. Since this submatrix of $\mathbf{A}^{\mathrm{T}}$ is just the transpose of the $(i,1)^{\text{th}}$ submatrix of **A**, by induction hypothesis, $M_{i1}^{\mathrm{T}} = M_{1i}$. Then,

$$\begin{aligned}\left|\mathbf{A}^{\mathrm{T}}\right| &= \sum_{i=1}^{n+1} a_{i1}^{\mathrm{T}} (-1)^{i+1} M_{1i} \\ &= \sum_{i=1}^{n+1} a_{1i} (-1)^{1+i} M_{1i}.\end{aligned}$$

The last term is just $\left|\mathbf{A}\right|$ as expanded along the first row.❑

The following theorem investigates the determinant of matrix when its rows are manipulated. By the last result **(g)**, we also have corresponding results for the row manipulations. Thus the results in **(d)**, **(e)** and **(f)** of the following corollary are just the determinant of a matrix after an elementary row operation is performed.

Corollary 4.1 Let **A** be a matrix in $\boldsymbol{R}^{n\times n}$ and $\mathbf{a}_{i\cdot}$ be its i^{th}row, so

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_{1\cdot} \\ \mathbf{a}_{2\cdot} \\ \vdots \\ \mathbf{a}_{n\cdot} \end{bmatrix}.$$

Then,

a) If $\mathbf{a}_{i\cdot} = \mathbf{0}$, $i = 1,2,\ldots,n$, then $\left|\mathbf{A}\right| = 0$.

b) If $\mathbf{a}_{i\cdot} = \mathbf{a}'_{i\cdot} + \mathbf{a}''_{i\cdot}$, $i = 1,2,\ldots,n$, then**,**

$$\begin{vmatrix} \mathbf{a}_{1\cdot} \\ \vdots \\ \mathbf{a}_{i\cdot} \\ \vdots \\ \mathbf{a}_{n\cdot} \end{vmatrix} = \begin{vmatrix} \mathbf{a}_{1\cdot} \\ \vdots \\ \mathbf{a}'_{i\cdot} \\ \vdots \\ \mathbf{a}_{n\cdot} \end{vmatrix} + \begin{vmatrix} \mathbf{a}_{1\cdot} \\ \vdots \\ \mathbf{a}''_{i\cdot} \\ \vdots \\ \mathbf{a}_{n\cdot} \end{vmatrix}.$$

c) For any scalar c and $j = 1,2,\ldots,n$, $\begin{vmatrix} \mathbf{a}_{1\cdot} \\ \vdots \\ c\mathbf{a}_{i\cdot} \\ \vdots \\ \mathbf{a}_{n\cdot} \end{vmatrix} = c\begin{vmatrix} \mathbf{a}_{1\cdot} \\ \vdots \\ \mathbf{a}_{i\cdot} \\ \vdots \\ \mathbf{a}_{n\cdot} \end{vmatrix}$.

d) If $\mathbf{a}_{i\cdot} = \mathbf{a}_{k\cdot}$ for some $i \neq k$, then $|\mathbf{A}| = 0$.

e) For any pair of rows i and k, $i \neq k$,

$$\begin{vmatrix} \mathbf{a}_{1\cdot} \\ \vdots \\ \mathbf{a}_{i\cdot} + c\mathbf{a}_{k\cdot} \\ \vdots \\ \mathbf{a}_{k\cdot} \\ \vdots \\ \mathbf{a}_{n\cdot} \end{vmatrix} = \begin{vmatrix} \mathbf{a}_{1\cdot} \\ \vdots \\ \mathbf{a}_{i\cdot} \\ \vdots \\ \mathbf{a}_{k\cdot} \\ \vdots \\ \mathbf{a}_{n\cdot} \end{vmatrix} = |\mathbf{A}|.$$

f) For any pair of rows i and k, $i \neq k$,

$$|\mathbf{A}| = \begin{vmatrix} \mathbf{a}_{1\cdot} \\ \vdots \\ \mathbf{a}_{i\cdot} \\ \vdots \\ \mathbf{a}_{k\cdot} \\ \vdots \\ \mathbf{a}_{n\cdot} \end{vmatrix} = -\begin{vmatrix} \mathbf{a}_{1\cdot} \\ \vdots \\ \mathbf{a}_{k\cdot} \\ \vdots \\ \mathbf{a}_{i\cdot} \\ \vdots \\ \mathbf{a}_{n\cdot} \end{vmatrix}.$$

Proof Follows directly from the last theorem by noting that $|\mathbf{A}^{\mathrm{T}}| = |\mathbf{A}|$.❑

Definition 4.3 A square matrix is ***nonsingular*** if its determinant is nonzero. Otherwise, it is called ***singular***.

Problem Simon & Blume [1994], page 734, #26.16, a-b. Prove the following results for matrices:

a. $|r\mathbf{A}| = r^n|\mathbf{A}|$;

b. $|-\mathbf{A}| = (-1)^n|\mathbf{A}|$.

Problem Johnson, Riess & Arnold [1998], page 246, #28. An $n \times n$ matrix $\mathbf{A}$ is called ***skew symmetric*** if $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$. Show that if $\mathbf{A}$ is skew symmetric, then $\left|\mathbf{A}\right| = \left(-1\right)^n \left|\mathbf{A}\right|$. Now, argue that the determinant of an $n \times n$ skew-symmetric matrix is zero when n is an odd integer.

Problem Show that for $1 \le m < k$, $1 \le n < k$,

$$\begin{vmatrix} \mathbf{A}_{m\times n} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_{(k-m)\times(k-n)} \end{vmatrix} = \begin{cases} \left|\mathbf{A}\right|\left|\mathbf{B}\right|, & \text{if } m = n, \\ 0, & \text{otherwise.} \end{cases}$$

4.3 Determinant of Product of Matrices

Lemma 4.1 The elementary matrix is nonsingular. That is the determinant of an elementary matrix is nonzero, where

a. If $\mathbf{E}_1$ is an elementary matrix obtained by multiplying a row of the identity matrix by a nonzero scalar c, then

$$\left|\mathbf{E}_1\right| = c.$$

b. If $\mathbf{E}_2$ is an elementary matrix obtained by interchanging the positions of any pair of rows of identity matrix, then

$$\left|\mathbf{E}_2\right| = -1.$$

c. If $\mathbf{E}_3$ is an elementary matrix obtained by multiplying a row of identity matrix by a scalar and then add it to another row, then

$$\left|\mathbf{E}_3\right| = 1.$$

Proof Follows directly from Corollary 4.1.❑

Theorem 4.3 Let $\mathbf{A}$ be a matrix in $\boldsymbol{R}^{n\times n}$. Let $\mathbf{E}$ be an elementary matrix also in $\boldsymbol{R}^{n\times n}$. Then,

$$\left|\mathbf{EA}\right| = \left|\mathbf{E}\right|\left|\mathbf{A}\right|.$$

Proof We will prove only for one kind of elementary row operations where a row is multiplied by a scalar c. So, we need to show

$$\left|\mathbf{EA}\right| = c\left|\mathbf{A}\right| = \left|\mathbf{E}\right|\left|\mathbf{A}\right|.$$

The first equality is due to part **(c)** of Corollary 4.1, and the second due to the part **(a)** of Lemma 4.1. The proofs for the other two kinds of elementary row operations use similar argument and are left as exercises.❑

Corollary 4.2 Let **A** be a matrix in $\boldsymbol{R}^{n\times n}$. Let **E** be an elementary matrix also in $\boldsymbol{R}^{n\times n}$. Then,

$$|\mathbf{AE}| = |\mathbf{A}||\mathbf{E}|.$$

Proof Use Theorem 4.2 above and problems in Chapter 3, page 10/11, about the elementary column operation and elementary matrix.❑

We are now ready to show that the determinant of the product of matrices is the product of the determinants.

Theorem 4.4 Let **A** and **B** be matrices in $\boldsymbol{R}^{n\times n}$. Then

$$|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}| = |\mathbf{BA}|.$$

Proof We need to show only the first equality. (Why?) Suppose **A** is full rank. By Corollary 3.2, there exists elementary matrices $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_k$, such that,

$$\mathbf{A} = \mathbf{F}_1\mathbf{F}_2\cdots\mathbf{F}_k.$$

By repeated applications of Theorem 4.3, we have

$$\begin{aligned}
|\mathbf{AB}| &= |\mathbf{F}_1\mathbf{F}_2\cdots\mathbf{F}_k\mathbf{B}| \\
&= |\mathbf{F}_1(\mathbf{F}_2\cdots\mathbf{F}_k\mathbf{B})| \\
&= |\mathbf{F}_1||\mathbf{F}_2\cdots\mathbf{F}_k\mathbf{B}| \\
&= |\mathbf{F}_1||\mathbf{F}_2|\cdots|\mathbf{F}_k||\mathbf{B}| \\
&= |\mathbf{F}_1\mathbf{F}_2\cdots\mathbf{F}_k||\mathbf{B}| \\
&= |\mathbf{A}||\mathbf{B}|.
\end{aligned}$$

Now suppose that **A** is not full rank. By Theorem 3.5, there exist elementary matrices $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_k$, such that, $\mathbf{A} = \mathbf{F}_1\mathbf{F}_2\ldots\mathbf{F}_k\mathbf{R}$, where **R** is a row echelon matrix. Since **A** is not full rank, **R** has at least one zero row. Thus, $|\mathbf{R}| = 0$ and

$$|\mathbf{A}| = |\mathbf{F}_1\mathbf{F}_2\cdots\mathbf{F}_k\mathbf{R}| = |\mathbf{F}_1\mathbf{F}_2\cdots\mathbf{F}_k||\mathbf{R}| = 0.$$

We will now show the theorem holds by showing that $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}| = 0$. The product $|\mathbf{A}||\mathbf{B}|$ is zero because $|\mathbf{A}| = 0$. By the same repeated applications of Theorem 4.3, the leftmost term is

$$\begin{aligned} |\mathbf{AB}| &= |\mathbf{F}_1\mathbf{F}_2\cdots\mathbf{F}_k\mathbf{RB}| \\ &= |\mathbf{F}_1\mathbf{F}_2\cdots\mathbf{F}_k||\mathbf{RB}|. \end{aligned}$$

Since **R** has at least one zero row, so does **RB** (See the problem in Chapter 3, page 3/11). Thus $|\mathbf{RB}| = 0$, and we have $|\mathbf{AB}| = 0$ as needed.❑

Problem Show that if **A** or **B** is not square, Theorem 4.4 does not apply.

Problem **Simon & Blume** [1994], page 734, #26.16, c-d. Prove the following results for matrices:

a. $|\mathbf{A}_1\mathbf{A}_2\cdots\mathbf{A}_r| = |\mathbf{A}_1||\mathbf{A}_2|\cdots|\mathbf{A}_r|$;

b. $|\mathbf{A}^k| = (|\mathbf{A}|)^k$ for positive integer *k*.

Problem Let **A**, **B** and **C** be square matrices of the same order. Show that if $\mathbf{AB} = \mathbf{C}$ and **C** is singular, then the matrix **A** or **B** must be singular.

Problem Is $|\mathbf{A} + \mathbf{B}| = |\mathbf{A}| + |\mathbf{B}|$?

4. Determinant and Rank

The proof of Theorem 4.4 also tells us that the rank of a square matrix and its determinant are related.

Theorem 4.5 Let **A** be a matrix in $\boldsymbol{R}^{n\times n}$. Then **A** is full rank if, and only if, **A** is nonsingular, i.e., $|\mathbf{A}| \neq 0$.

Proof As in the proof of previous theorem, if **A** is full rank, there exists elementary matrices $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_k$, such that,

$$\mathbf{A} = \mathbf{F}_1\mathbf{F}_2\cdots\mathbf{F}_k.$$

So $|\mathbf{A}| = |\mathbf{F}_1\mathbf{F}_2\cdots\mathbf{F}_k| = |\mathbf{F}_1||\mathbf{F}_2|\cdots|\mathbf{F}_k| \neq 0$ because by Lemma 4.1 each elementary matrix has nonzero determinant.

In reverse, if $|\mathbf{A}| \neq 0$, show that the matrix **A** is full rank. Suppose for a moment that **A** is not full rank, and we will try to derive some contradiction. If **A** is not full rank, there exist elementary matrices $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_k$, that

$$\mathbf{A} = \mathbf{F}_1 \mathbf{F}_2 \cdots \mathbf{F}_k \mathbf{R},$$

where **R** is a row echelon matrix with at least one row being zero. Then,

$$|\mathbf{A}| = |\mathbf{F}_1 \mathbf{F}_2 \cdots \mathbf{F}_k||\mathbf{R}| = 0,$$

as $|\mathbf{R}| = 0$. This is a contradiction. So **A** has to be full rank.❑

A square matrix is nonsingular, according to the last theorem, if, and only if, it is full rank. With this result, we can summarize the equivalence we obtain so far in the following two corollaries.

Corollary 4.3 Let **A** be a square matrix in $\boldsymbol{R}^{n\times n}$. The following statements are equivalent:

1. $|\mathbf{A}| \neq 0$, i.e., nonsingular.
2. rank $\mathbf{A}$ = rank $\mathbf{A}^{\mathrm{T}} = n$.
3. $\mathbf{Ax} = \mathbf{b}$ has a unique solution for any given vector **b**.

Proof Follows directly from Theorem 4.5 and Theorem 2.2 and the detail is left as exercise.❑

Corollary 4.4 Let **A** be a square matrix in $\boldsymbol{R}^{n\times n}$. The following are equivalent:

1. $|\mathbf{A}| = 0$, i.e., singular.
2. rank $\mathbf{A} < n$ and rank $\mathbf{A}^{\mathrm{T}} < n$.
3. $\mathbf{Ax} = \mathbf{0}$ has a nontrivial solution.
4. $\mathbf{Ax} = \mathbf{b}$ either has no solution or infinite number of solutions.

Proof Follows directly from the previous corollary.❑

It will be shown in Chapter 7, page 62, Corollary 7.6, that rank $\mathbf{A}$ = rank $\mathbf{A}^{\mathrm{T}}$ for any matrix $\mathbf{A} \in \boldsymbol{R}^{m\times n}$. That is, the matrix **A** may not be square or not full rank.

Problem Let **A** and **B** be $n \times n$ matrices. Show that the product **AB** is nonsingular if, and only if, both **A** and **B** are.

Problem Leon [1994], # 18 a, page 62.

18.Let **U** be an $n \times n$ upper triangular matrix with nonzero diagonal entries.

a. Explain why **U** must be nonsingular.

Definition 4.4 Matrices **A** and **B** are ***row equivalent*** if there exists a series of finite number of elementary matrices $\mathbf{E}_1, \mathbf{E}_2, \ldots, \mathbf{E}_k$ such that $\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_1\mathbf{A} = \mathbf{B}$.

Problem Write the definition of ***column equivalence***.

Problem Leon [1994], # 23, 24, page 63.

23.**(a)** Prove that **A** is row equivalent to **B** and **B** is row equivalent to **C**, then **A** is row equivalent to **C**.

(b) Prove that any two nonsingular $n \times n$ matrices are row equivalent.

24.Prove that **B** is row equivalent to **A** if and only if there exists a nonsingular matrix **M** such that $\mathbf{B} = \mathbf{MA}$.

Problem Fraleigh & Beauregard [1995], page 272, #35 a-d, f.

35.Let **A** be a square matrix. Mark each of the following True or False.

______**a)** The determinant of a square matrix is the product of the entries on its main diagonal.

______**b)** The determinant of an upper-triangular square matrix is the product of the entries on its main diagonal.

______**c)** The determinant of a lower-triangular square matrix is the product of the entries on its main diagonal.

______**d)** A square matrix is nonsingular if an only if its determinant is positive.