Macroeconomics

Lecture 7

Lucas's Model of Asset Prices

- This model uses the general equilibrium concept.
- It assumes that there are a large number of identical agents solving problem (1) – (2) , in which $y_t = 0$ for all t.

Lucas's Model of Asset Prices

- The only durable good in the economy is a set of trees which are in equal number to the number of people in the economy.
- Each period, each tree yields dividends in the amount d_t to its owner at the beginning of period t.
- Let p_t be the price of a tree in period t, measured in units of consumption goods per tree.

From the Euler equation (6) *that*

$$1 = \beta E_t\left(\frac{p_{t+1}+d_{t+1}}{p_t}\right)\frac{U'(c_{t+1})}{U'(c_t)} \qquad (10)$$

In equilibrium, we must have $c_t = d_t$*, substituting it into* (10),

$$p_t = E_t\beta\frac{U'(d_{t+1})}{U'(d_t)}\left(p_{t+1}+d_{t+1}\right) \qquad (11)$$

$$or, \quad p_t = E_t\beta\left[\frac{U'(d_{t+1})}{U'(d_t)}p_{t+1}\right] + E_t\beta\left[\frac{U'(d_{t+1})}{U'(d_t)}d_{t+1}\right],$$

$$= E_t\beta\left[\frac{U'(d_{t+1})}{U'(d_t)}\left\{E_{t+1}\beta\left[\frac{U'(d_{t+2})}{U'(d_{t+1})}p_{t+2}\right] + E_{t+1}\beta\left[\frac{U'(d_{t+2})}{U'(d_{t+1})}d_{t+2}\right]\right\}\right]$$

$$+ E_t\beta\left[\frac{U'(d_{t+1})}{U'(d_t)}d_{t+1}\right]$$

Using resursion on (11) *and the law of iterated expectations, we find that*

$$p_t = E_t\sum_{j=1}^{\infty}\beta^j\left\{\prod_{s=0}^{j-1}\frac{U'(d_{t+s+1})}{U'(d_{t+s})}\right\}d_{t+j}, \qquad or$$

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \left\{ \frac{U'(d_{t+j})}{U'(d_{t+j-1})} \frac{U'(d_{t+j-1})}{U'(d_{t+j-2})} \frac{U'(d_{t+j-2})}{U'(d_{t+j-3})} \cdots\cdots \frac{U'(d_{t+2})}{U'(d_{t+1})} \frac{U'(d_{t+1})}{U'(d_t)} \right\} d_{t+j},$$

or,

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{U'(d_{t+j})}{U'(d_t)} d_{t+j},$$

$$= \sum_{j=1}^{\infty} \beta^j E_t \left[\frac{U'(d_{t+j})}{U'(d_t)} d_{t+j} \right], \qquad (12)$$

Eq. (12) *is a generlization of* (9) *in which the share price is an expected discounted stream of dividends but with time-varying and stochastic discount rates* $\left(or,\ \frac{U'(d_{t+j})}{U'(d_t)} \right)$.

$$Let \qquad U\left(c_t\right) = \ln c_t \qquad\qquad (13)$$

Re*call* (12),

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \frac{U'\left(d_{t+j}\right)}{U'\left(d_t\right)} d_{t+j}$$

Substituting (13), *with* $c_t = d_t$, *the above equation,*

$$p_t = E_t \sum_{j=1}^{\infty} \beta^j \left(\frac{d_t}{d_{t+j}}\right) d_{t+j} = E_t \sum_{j=1}^{\infty} \beta^j \left(d_t\right) = \sum_{j=1}^{\infty} \beta^j E_t \left(d_t\right) = d_t \left(\sum_{j=1}^{\infty} \beta^j\right)$$

$$p_t = \frac{\beta}{1-\beta} d_t \qquad\qquad (14)$$

Eq (14) *is an asset – pricing function, a mapping of state of the economy,* d_t, int*o the price of a capital asset at t.*

Lucas's Asset-pricing function

d_t is assumed to be governed by a Markov process with a time-invariant transition probability distribution function given by

$$prob\left(d_{t+1} \leq x^{'} \mid d_t = x\right) = F\left(x^{'}, x\right) = \int_0^{x'} f\left(s, x\right) ds$$

The conditional expectation in the Euler equation (10) is defined with respect to this transition probability.

Lucas's Asset-pricing function

- **The household is to maximize**

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t) \qquad (15)$$

- **subject to (16).**

$$c_t + p_t s_{t+1} \leq (p_t + d_t) s_t \qquad (16)$$

Lucas's Asset-pricing function

- We need a law of motion for the stock price p_t to fully spell out constraint (16) for all t, and to make sure that the conditional expectation in (15) is well specified.

Lucas's Asset-pricing function

- Eq (16) states that during the time t to t+1, consumer's consumption at t and the value of his/her asset at the beginning period t+1 must not exceed the value of his/her asset at the beginning period t plus the dividend that will be realized at the beginning period t.

Lucas's Asset-pricing function

- From eq. (14), we have that the asset price p_t can be expressed as a function of state variable at t.
- So, let the price function be

$$p_t = h(d_t) \qquad (17)$$

Lucas's Asset-pricing function

- where h is continuous, bounded function defined on the current state d_t.
- Eq (17) and the transition law F (d′, d) defines the perceived law of motion for asset price p_t .
- Let V (s [h (x) + x]) be the value function for the problem when the consumer initially owns s trees, when the current dividend equal x, and when the current price of trees is h (x).

Bellman's equation is

$$V\left(s\left[h(x)+x\right]\right)=\max_{s'}\left\{U\left(s\left[h(x)+x\right]-h(x)s'\right)+\beta\int V\left(s'\left[h(x')+x'\right]\right)dF(x',x)\right\}$$

A prime denotes next – period values. Euler equation is

$$\frac{\partial U\left(s\left[h(x)+x\right]-h(x)s'\right)}{\partial\left(s\left[h(x)+x\right]-h(x)s'\right)}\frac{\partial\left(s\left[h(x)+x\right]-h(x)s'\right)}{\partial s'}$$

$$+\beta\int\frac{\partial V\left(s'\left[h(x')+x'\right]\right)}{\partial\left(s'\left[h(x')+x'\right]\right)}\frac{\partial\left(s'\left[h(x')+x'\right]\right)}{\partial s'}dF(x',x)=0$$

$$h(x)\frac{\partial U\left(s\left[h(x)+x\right]-h(x)s'\right)}{\partial\left(s\left[h(x)+x\right]-h(x)s'\right)}=\beta\int\left[h(x')+x'\right]\frac{\partial V\left(s'\left[h(x')+x'\right]\right)}{\partial\left(s'\left[h(x')+x'\right]\right)}dF(x',x)$$

Differentiating Bellman equation w.r.t. state variable, one has

$$\frac{\partial V\left(s\left[h(x)+x\right]\right)}{\partial\left(s\left[h(x)+x\right]\right)}=\frac{\partial U\left(s\left[h(x)+x\right]-h(x)s'\right)}{\partial\left(s\left[h(x)+x\right]-h(x)s'\right)}\frac{\partial\left(s\left[h(x)+x\right]-h(x)s'\right)}{\partial\left(s\left[h(x)+x\right]\right)}=\frac{\partial U(c)}{\partial c}$$

The Euler equation becomes

$$h(x)\frac{\partial U(c(x))}{\partial c(x)} = \beta\int\left[h(x')+x'\right]\frac{\partial U(c(x'))}{\partial c(x')}dF(x',x)$$

or

$$w(x) = \beta\int w(x')dF(x',x) + \beta\int x'\frac{\partial U\left[c(x')\right]}{\partial c(x')}dF(x',x) \qquad (18)$$

where $\quad w(x) = h(x)\dfrac{\partial U\left[c(x)\right]}{\partial c(x)}$

In equilibrium, $s = s' = 1$, *and* $c(x) = s\left[h(x)+x\right] - h(x)s' = x, \ (\because c_t = d_t)$

Substituting these into (18) *gives*

$$w(x) = \beta\int w(x')dF(x',x) + \beta\int x'\frac{\partial U(x')}{\partial x'}dF(x',x) \qquad (19)$$

$Since\ U(x)\ is\ known,\ once\ w(x)\ has\ been\ determined,$

$h(x)\ can\ be\ computed\ from\quad h(x) = \dfrac{w(x)}{\dfrac{\partial U(x)}{\partial x}}$

$Lucas\ assume\ U(0) = 0\ and\ U(c)\ is\ concave\ and\ is\ bounded\ by\ B.$

$The\ implication\ is\ that\ x\dfrac{\partial U(x)}{\partial x}\ is\ bounded\ by\ B.\ It\ follows\ that$

$$g(x) \equiv \beta \int x' \frac{\partial U(x')}{\partial x'} dF(x', x) \leq \beta B$$

$g(x)\ is\ continuous\ and\ bounded\ function.$

$Eq\ (19)\ can\ be\ rewritten\ as$

$$w(x) = \beta \int w(x') dF(x', x) + g(x) \qquad (20)$$

$Eq\ (20)\ has\ a\ unique\ continuous\ and\ bounded\ solution.$

$The\ solution\ of\ the\ functional\ equation\ (20)\ is\ approached\ by\ iteration$

$on\ w^j(x)\ defined\ by$

$$w^{j+1}(x) = \beta \int w^j(x') dF(x', x) + g(x) \qquad (21)$$

Starting from any initial continuous and bounded function $w^0(x)$.

Once the limiting function $w(x)$ *is known, the price function* $h(x)$

can be calculated as $\dfrac{w(x)}{\dfrac{\partial U(x)}{\partial x}}$.

Eq (21) *can be rewritten as*

$$h^{j+1}(x)\frac{\partial U(x)}{\partial x} = \beta \int h^j(x')\frac{\partial U(x')}{\partial x'} dF(x',x) + g(x) \qquad (22)$$

- **Eq (22) shows a mapping of a perceived pricing function h ^{j} (x) into an actual pricing function h^{j+1}(x).**
- **A rational expectations equilibrium is a fixed point of this mapping from perceived pricing functions to actual pricing functions.**

$$h^{j+1}(x) = \beta \int h^j(x') \frac{\partial U(x')/\partial x'}{\partial U(x)/\partial x} dF(x', x)$$

$$+ \beta \int x' \frac{\partial U(x')/\partial x'}{\partial U(x)/\partial x} dF(x', x)$$

$$h^{j+1}(x) = \beta \int \left\{ h^j(x') + x' \right\} \left\{ \frac{\partial U(x')/\partial x'}{\partial U(x)/\partial x} \right\} dF(x', x) \qquad (23)$$

$Eq\ (23)$ *is in the form as* $Eq(11)$,

$$p_t = \beta E_t \left[\left\{ p_{t+1} + d_{t+1} \right\} \left\{ \frac{\partial U(x')/\partial x'}{\partial U(x)/\partial x} \right\} | \, d_t \right]$$

$\because\ p_t = h(x_t) \cong h^{j+1}(x)$ *and* $F(x', x)$ *together define the perceived law of motion for the tree prices,* $h^j(x')$.

Rational Expectation Equilibrium

- A rational expectations equilibrium is a fixed point of the mapping from perceived pricing functions, $h^j(x)[\partial U(x)/\partial x]$, to actual pricing functions, $h^{j+1}(x)[\partial U(x')/\partial x']$. (See Brouwer fixed-point theorem from Varian (1992) p.320-322)