EE320 Chapter 9
Optimization under Equality Constraint

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1 Effect of constraint

Suppose that we have a function with one choice variable and we want to find its maximum/minimum point:

$$\max_{x} y = -x^2 + 10x + 4$$

$$\begin{aligned} \frac{dy}{dx} &= -2x + 10 = 0 \\ x^* &= 5 \\ y^* &= 29 \\ \frac{d^2y}{dx^2} &= -2 < 0 \quad \text{maximum} \end{aligned}$$

This method is called “unconstraint optimization.” Consider a different case which we impose a certain condition on a function:

$$\max_{x} y = -x^2 + 10x + 4 \text{ subject to } x = 0$$

We call this problem as a “constraint optimization,” when values of x are restricted.

x = 0$\Rightarrow$ y = 4

In this case, we set $x = 0$, so we call it "equality constraint." If instead, we have the problem subject to an inequality x $\geq$ 2, "inequality constraint."

ex. 3-dimension figure

Note maximum number of constraint = number of independent variable

ex.

$y = -x^2+10x+4$ $\Rightarrow$ 1 independent variable = 1 maximum constraint

$y = f(x_1, x_2)$ $\Rightarrow$ 2 ind variable = 2 maximum constraints

$y = f(x_1, x_2, \ldots, x_n)$ $\Rightarrow$ n ind variable = n maximum constraints

2 Constraint Optimization

Suppose that a consumer has the following utility function: $U = x_1x_2 + 2x_1$, given $p_1 = 4$ and $p_2 = 2$, this consumer's budget constraint is then: $4x_1 + 2x_2 = 60$. What will the optimal combination of goods x_1 and x_2 that give the highest utility subject to budget constraint be?

We can solve this problem using 2 approaches:

i) Substitution method: Substitute the constraint into objective function and solve for an optimal choices:

$$
\begin{aligned}
4x_1 + 2x_2 &= 60 \Rightarrow x_2 = 30 - 2x_1 \text{ substitute into utility function} \\
\Rightarrow U &= \\
&= \\
\underline{\text{FONC}} \qquad \frac{dU}{dx_1} &= \\
\underline{\text{SOSC}} \qquad \frac{d^2U}{dx_1^2} &=
\end{aligned}
$$

ii) Lagrange multiplier method: Convert a constraint optimization problem into an unconstraint optimization problem)

$\mathcal{L} =$

$\mathcal{L} =$ $\Rightarrow$ give the same answer

$\mathcal{L} = L(\lambda, x_1, x_2)$

$\underline{\text{FOC}}: \frac{\partial \mathcal{L}}{\partial \lambda} =$

$\frac{\partial \mathcal{L}}{\partial x_1} =$

$\frac{\partial \mathcal{L}}{\partial x_2} =$

3 unknowns and 3 equations solve for λ^*, x_1^*, x_2^*

3 Interpretation of Lagrange multiplier

Consider the problem $\max\limits_{x,y} f(x,y)$ subject to $g(x,y) = c$

Solutions are $x^*(c), y^*(c), f^*(c)$

$f^*(c)$ is optimal value function AND $\frac{d}{dc}f^*(c) = \lambda(c)$

Lagrange multiplier λ is the rate at which the optimal value of the objective function changes $\Delta f(x,y)$ with respect to changes in the constraint constant Δc.

In economic application, c denotes the available stock of some resources, while $f(x,y)$ denotes utility, profit, etc.

$df^*(c) = \lambda(c)dc$ is the change in utility or profit that can be obtained from dc units more of the resources.

Thus, λ is called a "shadow price" of the resource

ex. for cost minimization problem , $\lambda = \frac{dC}{dQ_0} =$ MC

utility max. $\lambda = \frac{dU}{dY_0} =$ MU of income

profit max. $\lambda = \frac{d\Pi}{dQ_0}$

output max. $\lambda = \frac{dQ}{dC_0}$

ex. Objective function: $z = xy$ subject to $x + y = 6$

$\mathcal{L} = xy + \lambda(6 - x - y)$

$\Rightarrow x^* = 3, y^* = 3, \lambda^* = 3, z^* = 9$

ex. Objective function: $f(x_1, x_2) = x_1^2 + x_2^2$ subject to $g(x_1, x_2) = x_1 + 4x_2 = 2$

$\mathcal{L} = x_1^2 + x_2^2 + \lambda(2 - x_1 - 4x_2)$

$\Rightarrow x_1^* = \frac{2}{17}, x_2^* = \frac{8}{17}$

4 Second-order conditions for constrained optimization

$$\begin{aligned}
\text{Recall the FONC for } L(x,y) &= f(x,y) + \lambda[c - g(x,y)] \\
L_x(x,y) &= f_x(x,y) - \lambda g_x(x,y) \\
L_y(x,y) &= f_y(x,y) - \lambda g_y(x,y) \\
\lambda &= \tfrac{f_x}{g_x} = \tfrac{f_y}{g_y}
\end{aligned}$$

Find SOSC

$$\begin{aligned}
L_{xx}(x,y) &= f_{xx} - \lambda g_{xx} = f_{xx} - \tfrac{f_y}{g_y} g_{xx} \\
L_{yy}(x,y) &= f_{yy} - \lambda g_{yy} = f_{yy} - \tfrac{f_y}{g_y} g_{yy} \\
L_{xy}(x,y) &= f_{xy} - \lambda g_{xy} = f_{xy} - \tfrac{f_y}{g_y} g_{xy}
\end{aligned}$$

Thus, $d^2 z$ can be written as

$$\begin{aligned}
d^2 z &= L_{xx} dx^2 + 2L_{xy} dx dy + L_{yy} dy^2 \\
&= \begin{bmatrix} dx & dy \end{bmatrix} \begin{bmatrix} L_{xx} & L_{xy} \\ L_{yx} & L_{yy} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}
\end{aligned}$$

SOSC conditions are:

Max of z : $d^2 z$ is negative definite ($d^2 z < 0$) , st. dg = 0

Min of z : $d^2 z$ is positive definite ($d^2 z > 0$) , st. dg = 0

where $dg = g_x dx + g_y dy = 0$

Define the determinant of the Bordered Hessian matrix as

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix}$$

or SOSC conditions are

Max: d^2z is negative definite st. $g_x dx + g_y dy = 0$ iff $|\bar{H}| > 0$

Min: d^2z is positive definite st. $g_x dx + g_y dy = 0$ iff $|\bar{H}| < 0$

<u>ex.</u> Opt. $U = x_1x_2 + 2x_1$ subject to $4x_1 + 2x_2 = 60$. Check whether it is max or min value.

$\mathcal{L} = x_1x_2 + 2x_1 + \lambda(60 - 4x_1 - 2x_2)$

<u>FONC</u>

$$\begin{aligned} \mathcal{L}_1 &= \\ \mathcal{L}_2 &= \\ \mathcal{L}_\lambda &= \end{aligned}$$

<u>SOSC</u>

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} =$$

$\therefore |\bar{H}| =$

$U^*(8, 14) = 128$ is a maximum utility subject to budget constraint.

ex. $f(x_1, x_2) = x_1^* + x_2^*$ subject to $g(x_1, x_2) = x_1 + 4x_2 = 2$ Find their critical points and check it is max or min.

SOSC $\quad |\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} =$

5 Economic Applications of Constrained Optimization

Application: Utility maximization Problem

Objective function: $\max\limits_{x_1, x_2} U = U(x_2, x_2)$

Subject to $Y_0 = p_1 x_1 + p_2 x_2$

$\mathcal{L} = U(x_1, x_2) + \lambda(y_0 - p_1 x_1 - p_2 x_2)$

FONC

$$
\begin{aligned}
L_1 &= u_1(x_1, x_2) - \lambda p_1 &= 0 \\
L_2 &= u_2(x_1, x_2) - \lambda p_2 &= 0 \\
L_\lambda &= y_0 - p_1 x_1 - p_2 x_2 &= 0
\end{aligned}
$$

$D(x_1^*, x_2^*)$ is such that $\frac{u_1(x_1,x_2)}{u_2(x_1,x_2)} = \frac{P_1}{P_2}$ and $Y_0 = p_1 x_1 + p_2 x_2$

$$
\text{from} \quad \left.\begin{aligned} \lambda &= \tfrac{u_1}{p_1} \\ \lambda &= \tfrac{u_2}{p_2} \end{aligned}\right\} \tfrac{u_1}{p_1} = \tfrac{u_2}{p_2} \quad \Rightarrow \quad \tfrac{u_1}{u_2} = \tfrac{p_1}{p_2}
$$

SOSC

$$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 \\ g_1 & L_{11} & L_{12} \\ g_2 & L_{21} & L_{22} \end{vmatrix} = \begin{vmatrix} 0 & p_1 & p_2 \\ p_1 & u_{11} & u_{21} \\ p_2 & u_{21} & u_{22} \end{vmatrix} = -p_1^2u_{11}^2 + 2p_1p_2u_{12} - p_2^2u_{11}^2$$

ex. $U = 50A + 200B - 0.5A^2 - 2.5B^2$. Find A,B that maximize the utility given

$$\left.\begin{array}{rcl} P_A &=& 10 \\ P_B &=& 5 \\ \bar{Y} &=& 490 \end{array}\right\} 490 = 10A + 5B$$

$\therefore$ max $U = 50A + 200B - 0.5A^2 - 2.5B^2$

subject to 10A + 5B = 490

$$\begin{array}{rcl} A^* &=& 30 \\ B^* &=& 38 \\ U^*(A^*, B^*) &=& 5,040 \end{array}$$

$|\bar{H}| = 525 > 0$ max.

Application: Expenditure Minimization Problem

Objective function $\min\limits_{x_1,x_2} E = p_1x_1 + p_2x_2$

Subject to $\bar{U} = U(x_1, x_2)$

$\mathcal{L} = p_1x_1 + p_2x_2 + \lambda(\bar{U} - U(x_1, x_2))$

FONC

$$\left.\begin{array}{rclcl} L_1 &=& p_1 - \lambda u_1(x_1, x_2) &=& 0 \\ L_2 &=& p_2 - \lambda u_2(x_1, x_2) &=& 0 \\ L_\lambda &=& \bar{u} - u(x_1, x_2) &=& 0 \end{array}\right\} \frac{u_1}{u_2} = \frac{p_1}{p_2} \text{ and } \bar{u} = u(x_1, x_2)$$

<u>SOSC</u> $|\bar{H}| < 0$ min.

<u>ex.</u> $U = 50A + 200B - 0.5A^2 - 2.5B^2$ and consumer would like to have a utility fixed at $\bar{u} = 5{,}040$. Find A and B that minimize the expenditure given that $P_A = 10$ and $P_B = 5$

$\therefore$ min $E = 10A + 5B$ subject to $5,040 = 50A + 200B - 0.5A^2 - 2.5B^2$

$$\Rightarrow \begin{aligned} A^* &= 30 \\ B^* &= 38 \\ E^* &= 490 \end{aligned}$$

Duality: Given that household wants to maximize his/her utility function given the budget constraint, that particular household can also minimize the expenditure given the desired level of utility. We claim that the solutions for expenditure minimization and utility maximization are the same. Let's prove this statement:

<u>ex.</u> $u = x_1 x_2$ given p_1, p_2, y_0

$\max_{x_1,x_2} u = x_1 x_2$ subject to $p_1 x_1 + p_2 x_2 = y_0$

$\mathcal{L} = x_1 x_2 + \lambda(y_0 - p_1 x_1 - p_2 x_2)$

FOC:

$$\begin{aligned} \mathcal{L}_1 &= x_2 - \lambda p_1 = 0 && \Rightarrow \ x_2 = \lambda p_1 \\ \mathcal{L}_2 &= x_1 - \lambda p_2 = 0 && \Rightarrow \ x_1 = \lambda p_2 \\ \mathcal{L}_\lambda &= p_1 x_1 + p_2 x_2 = y_0 \end{aligned}$$

$\therefore \frac{x_2}{x_1} = \frac{p_1}{p_2} \Rightarrow x_2 = \frac{p_1}{p_2} x_1$

$\Rightarrow p_1 x_1^* + p_2(\frac{p_1}{p_2} x_1^*) = y_0$

$$2p_1 x_1^* = y_0$$

$\Rightarrow x_1^* = \frac{I}{2p_1}$

$x_2^* = \frac{I}{2p_2}$

$\min\limits_{x_1,x_2} E = p_1x_1 + p_2x_2$

st. $x_1 + x_2 = \bar{u}$

$\mathcal{L} = p_1x_1 + p_2x_2 + \lambda(\bar{u} - x_1x_2)$

FOC:

$$\begin{array}{rcl} \mathcal{L}_1 & = & p_1 - \lambda x_2 = 0 \;\Rightarrow\; p_1 = \lambda x_2 \\ \mathcal{L}_2 & = & p_2 - \lambda x_1 = 0 \;\Rightarrow\; p_2 = \lambda x_1 \\ \mathcal{L}_\lambda & = & x_1x_2 = \bar{u} \end{array}$$

$\therefore \frac{x_2}{x_1} = \frac{p_1}{p_2} \Rightarrow x_2 = \frac{p_1}{p_2}x_1$

$\bar{u} = x_1^* \cdot (\frac{p_1}{p_2}x_1^*)$

$\Rightarrow x_1^* = \sqrt{(\frac{p_2}{p_1}\bar{u})}$

$x_2^* = \sqrt{(\frac{p_1}{p_2}\bar{u})}$

Application: Profit Maximization Problem

Objective function $\max\limits_{q_1,q_2} \Pi = p_1q_1 + p_2q_2 - c(q_1, q_2)$

Subject to $\bar{Q}_0 = q_1 + q_2$

$\mathcal{L} = p_1q_1 + p_2q_2 - c(q_1, q_2) + (\bar{Q} - q_1 - q_2)$

<u>FONC</u>

$$\left.\begin{array}{l} L_1 = p_1 - c_1(q_1, q_2) - \lambda = 0 \\ L_2 = p_2 - c_2(q_1, q_2) - \lambda = 0 \end{array}\right\} p_1 - c_1 = p_2 - c_2 \text{ and } \bar{Q} = q_1 + q_2$$

$L_\lambda = \bar{Q} - q_1 - q_2 = 0$

SOSC $|\bar{H}|>0$ max.

ex. Suppose that $p_1 = 80$, $p_2 = 100$, $TC = 100 + 0.1q_1^2 + 0.2q_2^2$ and $q_1 + q_2 = 325$. Find q_1 and q_2 that maximize profit.

$$\max_{q_1,q_2} = 80q_1 + 100q_2 - 100 - 0.1q_1^2 - 0.2q_2^2 \text{ subject to } q_1 + q_2 = 325$$

$$\Rightarrow \mathcal{L} = 80q_1 + 100q_2 - 100 - 0.1q_1^2 - 0.2q_2^2 + \lambda(325 - q_1 - q_2)$$

FONC

$$L_1 =$$

$$L_2 =$$

SOSC $\quad |\bar{H}| =$

Application: Output Maximization Problem

Objective function $\max_{K,L} Q = Q(K, L)$

Subject to $\bar{C} = wL + rK$

$$\mathcal{L} = Q(K, L) + \lambda(\bar{C} - wL - rK)$$

FONC

$$\left.\begin{aligned} L_K &= Q_K - \lambda r = 0 \\ L_L &= Q_L - \lambda w = 0 \end{aligned}\right\} \frac{Q_K}{Q_L} = \frac{r}{w}$$

$$L_\lambda = C_0 - wL - rK = 0$$

We can find $Q^*(K^*, L^*)$, K^* , L^*

<u>SOSC</u> Also check $|\bar{H}| = \begin{vmatrix} 0 & r & w \\ r & Q_{KK} & Q_{LK} \\ w & Q_{KL} & Q_{LL} \end{vmatrix}$

<u>ex.</u> Suppose Q = KL , w = 6 , r = 10 , $\bar{C}$ = 60. Find K and L that maximize output.

$\max_{K,L} Q = KL$ subject to $60 = 6L + 10K$

$\mathcal{L}$ = KL + λ (60-6L-10K) $\quad \Rightarrow \quad L^* = 5 \; K^* = 3$

$|\bar{H}| = 120 > 0 \quad$ max.

Application: Cost Minimization Problem

Objective function $\quad \min_{K,L} C = wL + rK$

Subject to $\bar{Q}$ = f(K,L)

$\mathcal{L} = wL + rK + \lambda(\bar{Q} - f(K, L))$

<u>FONC</u>

$$\left.\begin{array}{l} L_K = r - \lambda f_K = 0 \\ L_L = w - \lambda f_L = 0 \end{array}\right\} \frac{f_K}{f_L} = \frac{r}{w} \qquad \Rightarrow \qquad C^*(K^*, L^*) \ , \ K^* \ , \ L^*$$

$$L_\lambda = \bar{Q} - f(K, L) = 0$$

$$\underline{\text{SOSC}}\ |\bar{H}| = \begin{vmatrix} 0 & f_K & f_L \\ f_K & L_{KK} & L_{LK} \\ f_L & L_{KL} & L_{LL} \end{vmatrix} < 0 \quad \text{min.}$$

<u>ex.</u> Suppose Q = KL. A firm will produce 15 units of goods. Find K and L that minimize total cost given that w = 6 , r = 10.

Min C(w,r) 6L +10K

st. 15 = KL

$\mathcal{L} = 6L + 10K + \lambda(15 - KL)$

$$\Rightarrow \quad \begin{aligned} K^* &= 3 \\ L^* &= 5 \\ |\bar{H}| &= -60 < 0 \end{aligned}$$

6 n-choice Variable and Multi-Constraint Cases

6.1 One Constraint

Objective function z = $f(x_1, x_2, ..., x_n)$

Subject to $g(x_1, x_2, ..., x_n) = c$ (one constraint)

$\mathcal{L}(x_1, x_2, ..., x_n) = f(x_1, x_2, ..., x_n) + \lambda\Big[c - g(x_1, x_2, ..., x_n)\Big]$

<u>FONC</u>

$$\left.\begin{aligned} L_1 &= f_1 - \lambda g_1 = 0 \\ L_2 &= f_2 - \lambda g_2 = 0 \\ &\vdots \\ L_n &= f_n - \lambda g_n = 0 \\ L_\lambda &= c - g(x_1, x_2, ..., x_n) = 0 \end{aligned}\right\} \text{(n+1) conditions}$$

<u>SOSC</u> $$|\bar{H}| = \begin{vmatrix} 0 & g_1 & g_2 & \cdots & g_n \\ g_1 & L_{11} & L_{12} & \cdots & L_{1n} \\ g_2 & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n & L_{n1} & L_{n2} & \cdots & L_{nn} \end{vmatrix}_{(n+1)\times(n+1)}$$

d^2z is positive definite st. dg = 0 if and only if $|\bar{H}_2|, |\bar{H}_3|, ..., |\bar{H}_n| < 0$ always negative $\Rightarrow$ <u>Min</u>

d^2z is negative definite st. dg = 0 iff $|\bar{H}_2| > 0$,

$|\bar{H}_3| < 0$

$\vdots$

$(-1)^n|\bar{H}_n| > 0$ alternate signs $\Rightarrow$ <u>Max</u>

<u>ex.</u> min C = wL +iK + rT find K , L and T that minimize cost

st. $\bar{Q}$ = f(K,L,T)

$\mathcal{L} = wL + iK + rT + \lambda(\bar{Q} - f(K,L,T))$

6.2 k Equality Constraints

Objective function $z = f(x_1, x_2, ..., x_n)$ subject to constraints:

$$\begin{aligned} g^1(x_1, x_2, ..., x_n) &= c_1 \\ g^2(x_1, x_2, ..., x_n) &= c_2 \\ \vdots \qquad & \quad \vdots \\ g^k(x_1, x_2, ..., x_n) &= c_k \end{aligned}$$

$$\mathcal{L} = f(x_1, x_2, ..., x_n) + \lambda_1[c_1 - g^1(\cdot)] + \lambda_2[c_2 - g^2(\cdot)] + \cdots + +\lambda_k[c_k - g^k(\cdot)]$$

$$\mathcal{L} = f(x_1, x_2, ..., x_n) + \sum_{j=1}^{k} \lambda_j[c_j - g^j(x_1, x_2, ..., x_n)]$$

<u>FONC</u>

$$\left.\begin{aligned} \tfrac{\partial L}{\partial x_1} &= f_1 - \lambda_1 \tfrac{\partial g^1}{\partial x_1} - \lambda_2 \tfrac{\partial g^2}{\partial x_1} - \cdots - \lambda_k \tfrac{\partial g^k}{\partial x_1} = 0 \\ & \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\ \tfrac{\partial L}{\partial x_n} &= f_n - \lambda_1 \tfrac{\partial g^1}{\partial x_n} - \lambda_2 \tfrac{\partial g^2}{\partial x_n} - \cdots - \lambda_k \tfrac{\partial g^k}{\partial x_n} = 0 \end{aligned}\right\} \text{n equations}$$

$$\left.\begin{aligned} \tfrac{\partial L}{\partial \lambda_1} &= c_1 - g^1(x_1, x_2, ..., x_n) = 0 \\ & \qquad\qquad\qquad\qquad\qquad \vdots \\ \tfrac{\partial L}{\partial \lambda_k} &= c_k - g^k(x_1, x_2, ..., x_n) = 0 \end{aligned}\right\} \text{k equations}$$

<u>SOSC</u>

$$\bar{H} = \left|\begin{array}{ccccccccc} 0 & 0 & \cdots & 0 & | & g_1^1 & g_2^1 & \cdots & g_n^1 \\ 0 & 0 & \cdots & 0 & | & g_1^2 & g_2^2 & \cdots & g_n^2 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & | & g_1^k & g_2^k & \cdots & g_n^k \\ - & - & - & - & - & - & - & - & - \\ g_1^1 & g_2^1 & \cdots & g_n^1 & | & L_{11} & L_{12} & \cdots & L_{1n} \\ g_2^1 & g_2^2 & \cdots & g_2^k & | & L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & \vdots & \ddots & \vdots & | & \vdots & \vdots & \ddots & \vdots \\ g_n^1 & g_n^2 & \cdots & g_n^k & | & L_{n1} & L_{n2} & \cdots & L_{nn} \end{array}\right|_{(n+k)\times(n+k)}$$

Consider (n-k) bordered leading principal minors:

$|\bar{H}_{k+1}|, |\bar{H}_{k+2}|, \ldots, |\bar{H}_n|$

Max of z, sign $|\bar{H}_{k+1}| = (-1)^{k+1}$ alternate sign for $|\bar{H}_{k+2}|$ and so on.

Min of z, sign $|\bar{H}_k| = (-1)^{k+1}$ take the same sign

<u>ex.</u> $z = f(x_1, x_2, x_3)$

st. $g(x_1, x_2, x_3) = c$ and $h(x_1, x_2, x_3) = d$

$\mathcal{L} = f(x_1, x_2, x_3) + \lambda_1[c - g] + \lambda_2[d - h]$

<u>FONC</u>:

<u>SOSC</u>: n = 3, k= 2 $\Rightarrow$ n-k = 1 only one bordered Hessian matrix

$|\bar{H}_n| = |\bar{H}_3|$ or $|\bar{H}_{k+1}| = |\bar{H}_{2+1}| = |\bar{H}_n|$

$$\therefore |\bar{H}_3| = \begin{vmatrix} 0 & 0 & g_1 & g_2 & g_3 \\ 0 & 0 & h_1 & h_2 & h_3 \\ g_1 & h_1 & L_{11} & L_{12} & L_{13} \\ g_2 & h_2 & L_{21} & L_{22} & L_{23} \\ g_3 & h_3 & L_{31} & L_{32} & L_{33} \end{vmatrix}_{5\times 5}$$

max if $|\bar{H}_{k+1}| = |\bar{H}_3| < 0$ because sign of $|\bar{H}_3| = (-1)^{2+1} = -1 < 0$ should be negative

max if $|\bar{H}_{k+1}| = |\bar{H}_3| > 0$ because sign of $|\bar{H}_3| = (-1)^2 = 1 > 0$ should be positive

<u>ex.</u> $z = f(x_1, x_2, x_3, x_4)$

st. $g(x_1, x_2, x_3, x_4) = c$

$h(x_1, x_2, x_3, x_4) = d$

<u>FONC</u>:

<u>SOSC</u> n = 4, k =2 $\Rightarrow$ n - k = 2 bordered Hessian matrix.

Consider $|\bar{H}_{k+1}| = |\bar{H}_{2+1}| = |\bar{H}_3| = \begin{vmatrix} 0 & 0 & g_1 & g_2 & g_3 \\ 0 & 0 & h_1 & h_2 & h_3 \\ g_1 & h_1 & L_{11} & L_{12} & L_{13} \\ g_2 & h_2 & L_{21} & L_{22} & L_{23} \\ g_3 & h_3 & L_{31} & L_{32} & L_{33} \end{vmatrix}$

$$|\bar{H}_{k+2}| = |\bar{H}_{2+2}| = |\bar{H}_4| = \begin{vmatrix} 0 & 0 & g_1 & g_2 & g_3 & g_4 \\ 0 & 0 & h_1 & h_2 & h_3 & h_4 \\ g_1 & h_1 & L_{11} & L_{12} & L_{13} & L_{14} \\ g_2 & h_2 & L_{21} & L_{22} & L_{23} & L_{24} \\ g_3 & h_3 & L_{31} & L_{32} & L_{33} & L_{34} \\ g_4 & h_4 & L_{41} & L_{42} & L_{43} & L_{44} \end{vmatrix}_{6\times 6}$$

max if sign of $|\bar{H}_{k+1}| = |\bar{H}_{2+1}| = |\bar{H}_3| = (-1)^{2+1} = -1$ negative or $|\bar{H}_3| < 0$ and $|\bar{H}_{k+2}| = |\bar{H}_4| > 0$

min if sign of $|\bar{H}_3| = (-1)^2 = 1$ or $|\bar{H}_3|>0$ and $|\bar{H}_{k+2}| = |\bar{H}_4| > 0$

ex. Given objective function $y = f(x_1, \dots, x_{35})$ and 22 constraints

1. What is the dimension of $\bar{H}$?

2. What is the dimension of zero matrix inside $\bar{H}$?

3. Write SOCs for max.

7 Other Economic Application

7.1 Homogeneous function

$F(\lambda x, \lambda y) = \lambda^r F(x, y)$

where r is a degree of homogeneous function.

r can indicate 3 types of return to scales (IRTS, CRTS, DRTS)

ex. Q = KL

K_0, $L_0 \quad \Rightarrow \quad Q_0 = K_0 L_0$

$\lambda K_0, \lambda L_0 \;\Rightarrow\; Q_1 = \lambda K_0 \lambda L_0 = \lambda^2 K_0 L_0 = \lambda^2 Q_0$

If r = 2, then if we increase both L and K by 2 times, then the level of output would be rising by $\lambda^2 = 2^2 = 4$ times = IRTS ex. Q =K+L

$$Q(\lambda K, \lambda L) = \lambda K_0 + \lambda L_0 = \lambda(K+L) = \lambda Q_0(K,L) \;\Rightarrow\; \text{CRTS}$$

ex. $Q = K^\alpha L^\beta$ What should the value of α abd β be such that Q has a property of CRTS?

$Q(K,L) = K^\alpha L^\beta$

$$\begin{aligned} Q(\lambda K, \lambda L) &= (\lambda K)^\alpha (\lambda L)^\beta \\ &= \lambda^{\alpha+\beta} K^\alpha L^\beta \\ &= \lambda^{\alpha+\beta} Q(K,L) \end{aligned}$$

$\alpha + \beta < 1$ DRTS
$\alpha + \beta = 1$ CRTS
$\alpha + \beta > 1$ IRTS

Recall $AC = \frac{TC}{Q}$

at IRTS, $\uparrow$ TC by 2, bit $\uparrow$ Q $> 2 \;\Rightarrow\;$ AC $\downarrow$

DRTS, $\uparrow$ TC by 2, bit $\uparrow$ Q $< 2 \;\Rightarrow\;$ AC $\uparrow$

<u>ex</u> $TC = 2\sqrt{Q}$

AC $= \frac{TC}{Q} = \frac{2\sqrt{Q}}{Q} = \frac{2}{\sqrt{Q}}$ IRTS $\therefore Q \uparrow \Rightarrow AC \downarrow$

$$\therefore F(\lambda x, \lambda y) = \lambda^r F(x,y) \quad \text{if} \begin{array}{llll} r = 1 & , & CRTS & \Rightarrow & \text{AC constant} \\ r > 1 & , & \text{IRTS} & \Rightarrow & \text{AC} \downarrow \\ r < 1 & , & \text{DRTS} & \Rightarrow & \text{AC} \uparrow \end{array}$$

7.2 Types of Goods

<u>ex.</u>

$$\begin{aligned} U &= x + y \\ P_x &= 2 \\ P_y &= 1 \\ I &= 100 \end{aligned}$$

(1st approach): $\mathcal{L} = x + y + \lambda(100 - 2x - y)$

<u>FONC</u>

$$\left.\begin{aligned} \mathcal{L}_\lambda &= 100 - 2x - y &= 0 \\ \mathcal{L}_x &= 1 - 2\lambda &= 0 \\ \mathcal{L}_y &= 1 - \lambda &= 0 \end{aligned}\right\} \text{Lagragian cannot solve the corner solution.}$$

(2nd approach)

$$\begin{aligned} MU_x &= 1 \\ MU_y &= 1 \\ MRS &= 1 \end{aligned}$$

$\therefore$ A consumer should buy more Y $\because$ it is cheaper and give same MU

$\Rightarrow$ Demand for x = 0 and Demand for y = 100

This case: $\frac{P_x}{P_y} = \frac{2}{1} = 1 > |MRS|$

How about $|\ \frac{P_x}{P_y}\ | < |MRS|$

$|\ \frac{P_x}{P_y}\ | = |MRS|$

ex. U = min [x,y]

x	y	u
1	1	1
1	2	1
2	1	1
2	3	2

"perfect complement"

ex. U = min [2x,y]

kink point is 2x = y

ex.
$$\left.\begin{array}{rcl} U & = & \text{min[x,y]} \\ P_x & = & 2 \\ P_y & = & 1 \\ I & = & 100 \end{array}\right\}$$

(1st approach) $\mathcal{L} = \min[x, y] + \lambda(100 - 2x - y)$

Solve for $x^*, \; y^*$

(2nd approach)

kink point x = y and budget constraint: $100 - 2x - y = 0$

Reference

Aroonruengsawat, Anin. EE320 Lecture Handouts.

Chiang, A. C. and Wainwright, K. (2005) "Fundamental Methods of Mathematical Economics," 4th edition, McGraw-Hill, Inc., Singapore.

Dowling, E. T. (2001) "Schaums Outline of Theory and Problem of Introduction to Mathematical Economics", 3rd edition, The McGraw-Hill Companies, Inc.

Holden, K. and Pearson, A.W. (1992) "Introductory Mathematics for Economics and Business," Second edition, The Macmillan Press Ltd.

Sydsaeter, K. and P. Hammond. (2006) "Essential Mathematics for Economic Analysis," 2nd edition, Prentice Hall.