$$\begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

Solve $\underline{R}\,\underline{x} = 0$, let $\left.\begin{array}{l} x_3 = c_3 \;;\; c_3 \in \mathbb{R} \\ x_4 = c_4 \;;\; c_4 \in \mathbb{R} \end{array}\right\}$ <constant values>

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & 0 \\ 0 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} ;\; c_3, c_4 \in \mathbb{R}$$

$$\underline{x} = x_3 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} ;\; \begin{array}{l} c_3 \in \mathbb{R} \\ c_4 \in \mathbb{R} \end{array}$$

If swapping columns to group x pivot together.

$$\left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 2 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_3 \\ \hline x_2 \\ x_4 \end{array}\right] = 0 \;-$$

Solution:

$$\left[\begin{array}{c} x_1 \\ x_3 \\ \hline x_2 \\ x_4 \end{array}\right] = \left[\begin{array}{cc} -2 & 2 \\ 0 & -2 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} ;\; c_3, c_4 \in \mathbb{R}$$

$\therefore$ We can rewrite the solutions as

$$\begin{bmatrix} x_1 \\ x_3 \\ \hline x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} \underline{x}_{pivot} \\ \hdashline \underline{x}_{free} \end{bmatrix} = \begin{bmatrix} -\underline{F} \\ \hdashline \underline{I} \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} ; \begin{matrix} c_3 \in \mathbb{R} \\ c_4 \in \mathbb{R} \end{matrix}$$

If we consider

$$\underline{R}_{gen.} = \begin{array}{c} \begin{matrix} \quad \underline{I} & \quad \underline{F} \end{matrix} \\ \left[\begin{array}{cc|cc} 1 & 0 & 2 & -2 \\ 0 & 1 & 0 & 2 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] \end{array} = \left[\begin{array}{c|c} \underline{I} & \underline{F} \\ \hline 0 & 0 \end{array}\right]$$

$$\underline{R}\,\underline{N} = 0$$

Let $\underline{N}$ be a matrix that each column is a solution to $\underline{R}\,\underline{x} = 0$.

$$\therefore \quad \underline{R}\,\underline{N} = \begin{bmatrix} \underline{I} & \underline{F} \\ \underline{0} & 0 \end{bmatrix} \begin{bmatrix} -\underline{F} \\ \underline{I} \end{bmatrix} = 0$$

$$\therefore \quad \underline{N} = \begin{bmatrix} -\underline{F} \\ \underline{I} \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 0 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A linear combination of column of $\underline{N}$ forms all solutions to $\underline{R}\,\underline{x} = 0$.

$$\underline{X} = \begin{bmatrix} -2 & 2 \\ 0 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} C_3 \\ C_4 \end{bmatrix} \quad ; \quad C_3 \in \mathbb{R},\ C_4 \in \mathbb{R}.$$

$$\underline{X} = C_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + C_4 \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} \quad ; \quad C_3 \in \mathbb{R},\ C_4 \in \mathbb{R}$$

∴ If we have RREF of $\underline{A}$ (ie. $\underline{R}$) and we would like to find solution to $\underline{A}\underline{x} = 0$ (ie. $\underline{R}\underline{x} = 0$) (we are solving a homogeneous system).

we can obtain solution directly by arranging $\underline{R} \rightarrow \underline{R}_{gen}$.

$$\underline{R}_{gen} = \left[\begin{array}{c|c} \underline{I}_{part} & \underline{F}_{p \times r} \\ \hline 0 & 0 \end{array}\right].$$

and we can obtain answer directly from $\underline{R}_{gen}$

$$\underline{X} = \begin{bmatrix} -\underline{F} \\ \underline{I} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \quad ; C_1, C_2 \in \mathbb{R}.$$

# of free variables

If : 1 free variables

$$\underline{X} = \begin{bmatrix} -\underline{F} \\ \underline{\underline{I}} \end{bmatrix} C_1 \quad ; C_1 \in \mathbb{R}.$$

Note: $\underline{I} = [1]_{1\times 1}$

If : 3 free variables

$$\underline{X} = \begin{bmatrix} -\underline{F} \\ \underline{I} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \quad ; \begin{matrix} C_1 \in \mathbb{R} \\ C_2 \in \mathbb{R} \\ C_3 \in \mathbb{R} \end{matrix}$$

Note: $\underline{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

example :

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 6 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{array}{c} \begin{array}{ccc} & \underline{I} & \quad \underline{F} \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hline 0 & 0 & 0 \end{array}\right] \end{array} \Rightarrow \left[\begin{array}{c|c} \underline{I} & \underline{F} \\ \hline 0 & 0 \end{array}\right] \quad \therefore \text{ soln}$$

$$\underline{X} = \begin{bmatrix} -\underline{F} \\ \hline \underline{I} \end{bmatrix} C_1 \; ; C_1 \in \mathbb{R}$$

$\underline{X} = [-1$