OLS-GLS (Matrix Approach)

16 มกราคม 2561 8:03

$$Y_i = \beta_1 + \beta_2 X_{2i} + \beta_3 X_{3i} + \cdots \beta_k X_{ki} + u_i \qquad i = 1, 2, \ldots, n$$

$$\begin{array}{ll}
i=1 & Y_1 = \beta_1 + \beta_2 X_{21} + \beta_3 X_{31} + \cdots \beta_k X_{k1} + u_1 \\
i=2 & Y_2 = \beta_1 + \beta_2 X_{22} + \beta_3 X_{32} + \cdots + \beta_k X_{k2} + u_2 \\
\vdots & \quad \vdots \\
i=n & Y_n = \beta_1 + \beta_2 X_{2n} + \beta_3 X_{3n} + \cdots + \beta_k X_{kn} + u_n
\end{array}$$

$$\underset{n\times 1}{\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix}} = \underset{n\times k}{\begin{bmatrix} 1 & X_{21} & X_{31} & \cdots & X_{k1} \\ 1 & X_{22} & X_{32} & \cdots & X_{k2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & X_{2n} & X_{3n} & \cdots & X_{kn} \end{bmatrix}} \underset{k\times 1}{\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \vdots \\ \beta_k \end{bmatrix}} + \underset{n\times 1}{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}}$$

$$\underset{n\times 1}{Y} = \underset{n\times k}{X}\,\underset{k\times 1}{\hat{\beta}} + \underset{n\times 1}{\hat{u}} \quad \Rightarrow \quad \hat{u} = Y - X\hat{\beta}$$

$$\underset{1\times n}{\hat{u}'}\,\underset{n\times 1}{\hat{u}} = [\hat{u}_1 \ \hat{u}_2 \ \cdots \ \hat{u}_n] \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_n \end{bmatrix} = \hat{u}_1^2 + \hat{u}_2^2 + \cdots + \hat{u}_n^2 = \sum_{i=1}^{n} \hat{u}_i^2$$

RSS

$$(Y - X\hat{\beta})'(Y - X\hat{\beta}) = Y'Y - \underset{1\times n}{Y'}\,\underset{n\times k}{X}\,\underset{k\times 1}{\hat{\beta}} - \hat{\beta}'X'Y + \hat{\beta}'X'X\hat{\beta}$$

$$\frac{\partial \hat{u}'\hat{u}}{\partial \hat{\beta}} = \frac{\partial\left(Y'Y - 2\hat{\beta}'X'Y + \hat{\beta}'X'X\hat{\beta}\right)}{\partial \hat{\beta}} = 0$$

$$= 0 - 2X'Y + 2X'X\hat{\beta} = 0$$

$$\cancel{2}X'X\hat{\beta} = \cancel{2}X'Y$$

$$\cancel{(X'X)^{-1}X'X}\hat{\beta} = (X'X)^{-1}X'Y$$

$$\cancel{(X'X)^{-1}}\,\cancel{X'X}\hat\beta = (X'X)^{-1}X'Y$$

$$\boxed{\underset{k\times 1}{\hat\beta_{OLS}} = (\underset{k\times n}{X'}\underset{n\times k}{X})^{-1}\underset{k\times n}{X'}\underset{n\times 1}{Y}}$$

$$u\cdot u' = \underset{n\times 1}{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}} \underset{1\times n}{\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}}$$

$E[(X_i - E(X))(Y_i - E(Y))] = cov(X,Y)$

$E[(X_i - E(X))^2] = var X$

$E(u_i) = 0$

$E[(u_1 - E(u_1))^2]$

$$E(u\cdot u') = \begin{bmatrix} E(u_1^2) & E(u_1u_2) & \cdots & E(u_1u_n) \\ E(u_2u_1) & E(u_2^2) & \cdots & E(u_2u_n) \\ \vdots & \vdots & \ddots & \vdots \\ E(u_nu_1) & E(u_nu_2) & \cdots & E(u_n^2) \end{bmatrix}$$

$$\underset{n\times n}{\Sigma} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_n^2 \end{bmatrix} = \underset{n\times n}{\begin{bmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{bmatrix}}$$

$=0$

$$\Sigma = \sigma^2 \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \sigma^2 \underset{n\times n}{I}$$

$$\hat\beta_{GLS} = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y$$

OLS

$\Sigma = \sigma^2 I$

$$\beta_{GLS} = (X \Sigma X) X \Sigma Y$$

$$= \left(X'(\sigma^2 I)^{-1} X\right)^{-1} X'(\sigma^2 I)^{-1} Y$$

$$= \cancel{\sigma^2}\frac{1}{\cancel{\sigma^2}}(X'X)^{-1}X'Y$$

$$\hat{\beta}_{OLS} = (X'X)^{-1}X'Y$$