EE320 Chapter 6
Optimization without Constraints: One Independent Variable Case

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1 Introduction

Optimal value in economics is derived either from

- maximizing some economic variables such as profit, utility or
- minimizing some economic variables such as cost, budget

Duality.

by looking at the objective function (dependent variable)

and making choice (independent variable)

Example. Firm's problem:

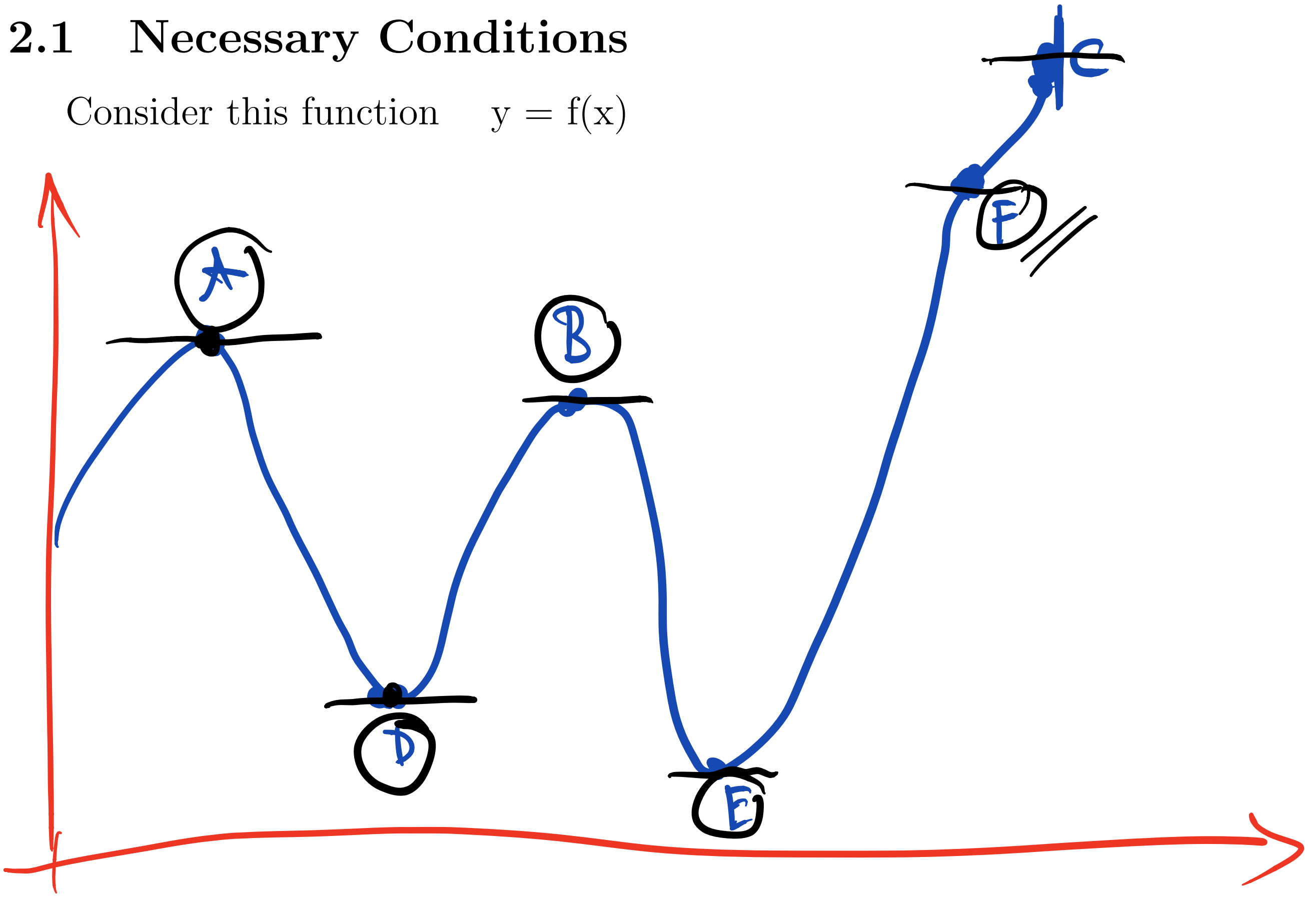
$$\Pi(Q) \equiv R(Q) - C(Q)$$

where Π is the object of optimization/maximization.
Q is the choice variable.

2 Maxima, Minima and Inflection point

2.1 Necessary Conditions

Consider this function y = f(x)



$$\begin{aligned} A, B &= \text{relative maxima} \\ D &= \text{relative minima} \\ C &= \text{absolute maxima} \\ E &= \text{absolute minima} \\ F &= \text{inflection point} \end{aligned}$$

To test that point, find 1^{st} - order derivative

$$\left.\begin{aligned} \left.\frac{dy}{dx}\right|_A = 0 \\ \left.\frac{dy}{dx}\right|_B = 0 \end{aligned}\right\} \text{relative max} \qquad \left.\begin{aligned} \left.\frac{dy}{dx}\right|_D = 0 \\ \left.\frac{dy}{dx}\right|_E = 0 \end{aligned}\right\} \text{relative min}$$

$$\left.\left.\frac{dy}{dx}\right|_F = 0 \right\} \text{inflection point}$$

These are "necessary condition" for relative maximum/minimum. We still cannot indicate whether it is maximum, minimum or inflection point)

Example: $y = x^3 - 12x^2 + 36x + 8$ $\Rightarrow$ $x^2 - 8x + 12 = 0$

First-order condition: $\frac{dy}{dx} = 3x^2 - 24x + 36 = 0$ $\quad x^* = 2, 6$

Find tangent line that slope is zero $\Rightarrow (x-2)(x-6) = 0$

2.2 Sufficient Condition (2^{nd} order derivative)

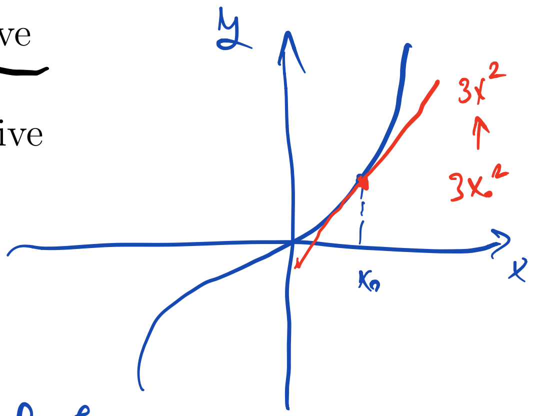
Suppose $\boxed{y = f(x)}$

$\frac{d^2y}{dx^2}$ $\quad \cancel{\frac{dy^2}{dx^2}}$

$\frac{dy}{dx} = y' = f'(x)$ $\quad$ 1^{st} order derivative

$\frac{d^2y}{dx^2} = y'' = f''(x)$ $\quad$ 2^{nd} order derivative

Note that $\boxed{\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}}$ $\quad \frac{d}{dx}\left(\frac{d^2x}{dx^2}\right) = \frac{d^3x}{dx^3}$



slope of f(x)

Example: $y = x^3$ ✓

$\frac{dy}{dx} = 3x^2$

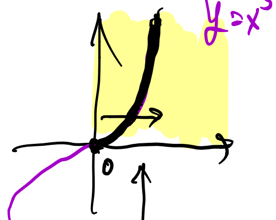
$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx} 3x^2 = 6x$: slope of the tangent line.

$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}(6x) = 6$

For both $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} > 0$ means the slope is positive and is increasing.

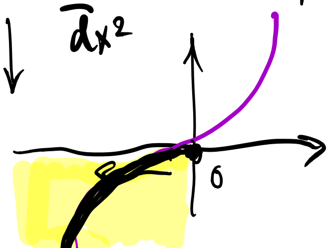
$\boxed{y = x^3}$

$\frac{dy}{dx} = 3x^2$: $\forall x, \frac{dy}{dx} > 0$.



$\frac{d^2y}{dx^2} = 6x$: If $x \geq 0$, $\frac{d^2y}{dx^2} \geq 0$ slope ↗

If $x < 0$, $\frac{d^2y}{dx^2} < 0$ slope ↘



∴ Interpretation of 1^{st} and 2^{nd} derivative :

1^{st}derivative $f'(x) > 0$ the value of the function tends to ....↑.....

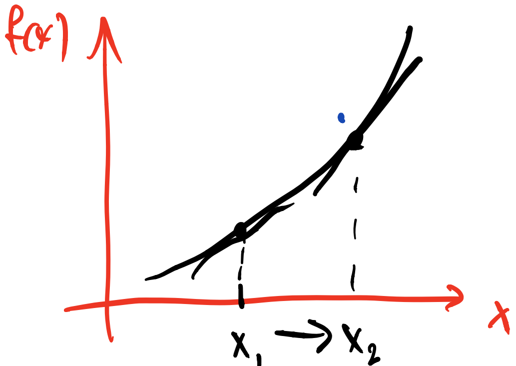
$f'(x) < 0$ the value of the function tends to ....↓.....

2^{nd}derivative $f''(x) > 0$ the slope of the function tends to ....↑.....

$f''(x) < 0$ the slope of the function tends to ....↓.....

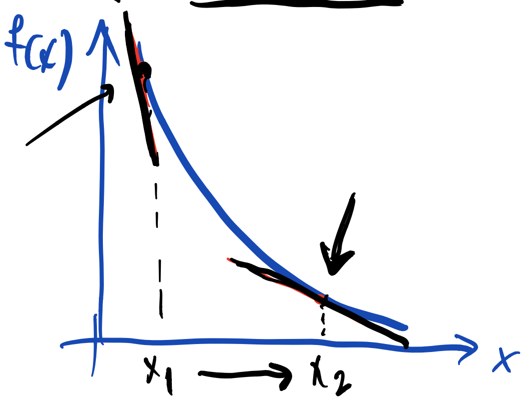
Possible cases

f(x) 1) $f'(x_0) > 0$ and $f''(x_0) > 0$ slope is ...(+)... and ...↑... as x increases.



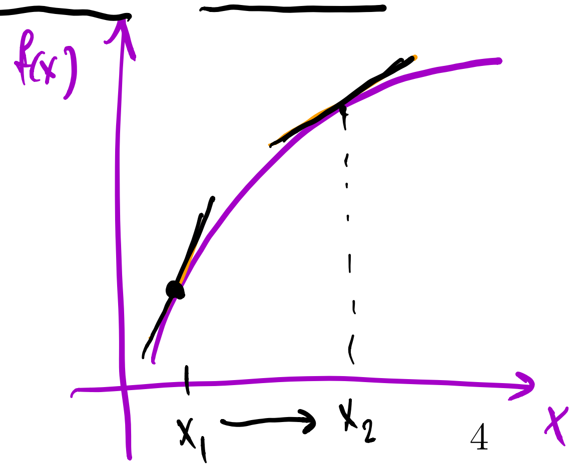
x ↑ ⇒ slope becomes steeper.

f(x) 2) $f'(x_0) < 0$ and $f''(x_0) > 0$ slope is ...(−)... and ...↑... as x increases.



x ↑ ⇒ slope becomes flatter.

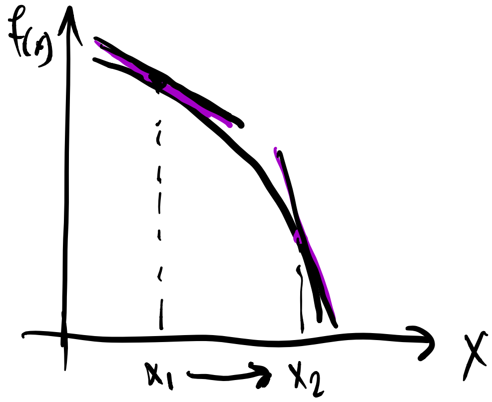
f(x) 3) $f'(x_0) > 0$ and $f''(x_0) < 0$ slope is ...(+)... and ...↓... as x increases.



x ↑ ⇒ slope becomes flatter.

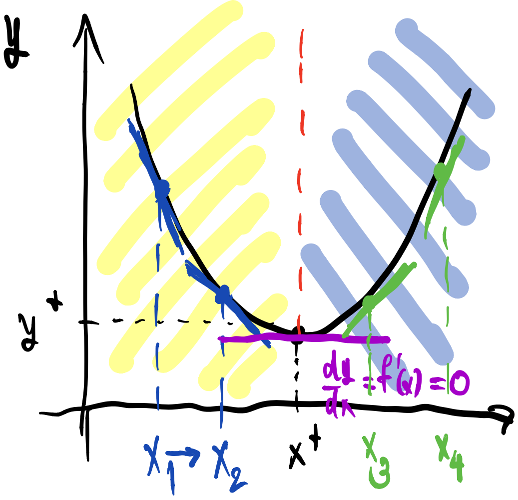
ex. diminish marginal product.

$f(x)$ 4) $\underline{f'(x_0) < 0}$ and $\underline{f''(x_0) < 0}$ slope is ..$\ominus$.. and ..$\downarrow$.. as x increases.



$x \uparrow \Rightarrow$ slope becomes steeper

Example:

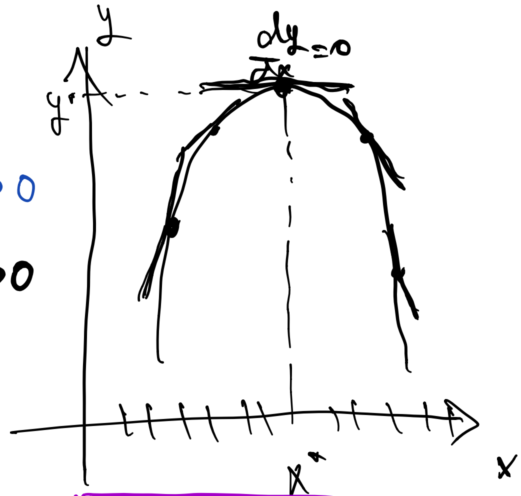


minimum at y^*

If $x < x^*$; $\underline{f'(x_1) < 0}$, $f''(x_1) > 0$

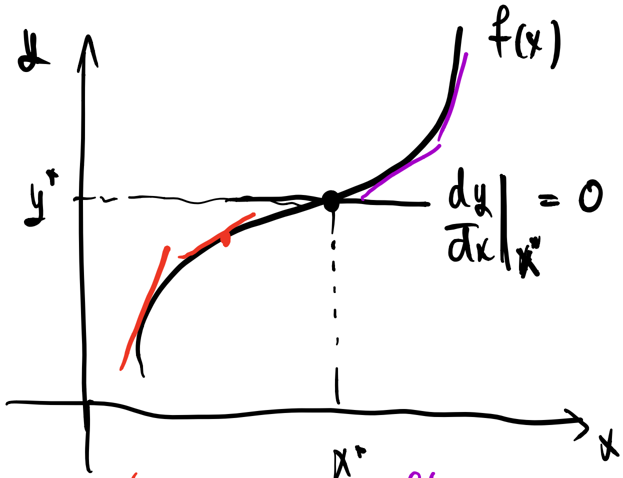
If $x > x^*$; $f'(x_3) > 0$, $f''(x_3) > 0$

$\boxed{\forall x,\ f''(x) > 0}$



$\boxed{\forall x,\ f''(x) < 0}$

Note inflection point: $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$



$f'(x) > 0$ | $f'(x) > 0$
$\boxed{f''(x) < 0}$ * | $\boxed{f''(x) > 0}$ *
✓ | ✓

min : $f''(x) > 0$

max : $f''(x) < 0$

inflection : $f''(x) = 0$.

3 Convexity and Concavity

$f''(x) < 0$: decreasing

$f''(x) > 0$: increasing

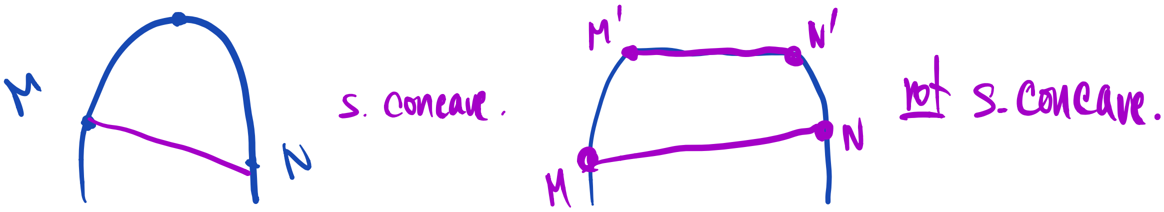
lines.

Definition: $\boxed{\text{Strictly convex}}$

A function f(x) is strictly convex if we pick any pair of points M and N on its curve and join them by a straight line, the line segment MN must lie entirely above the curve, except at points M and N.

Definition: $\boxed{\text{Strictly concave}}$

A function f(x) is strictly concave if we pick any pair of points M and N on its curve and join them by a straight line, the line segment MN must lie entirely below the curve, except at points M and N.

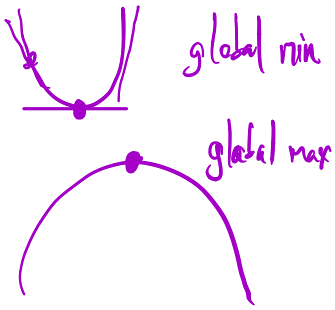


equations

Or assume that $\boxed{\text{f(x) is continuous and twice differentiable:}}$

If $f''(x) > 0 \;\; \forall X$, then f(x) is strictly convex function.

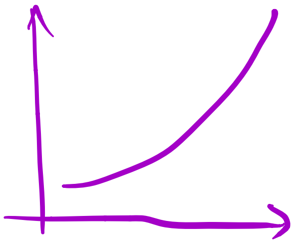
If $f''(x) < 0 \;\; \forall X$, then f(x) is strictly concave function.

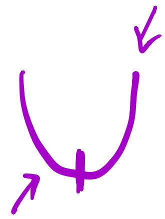


Example: 1.Increasing convex

$f'(x) > 0$ $\forall x$

$f''(x) > 0$

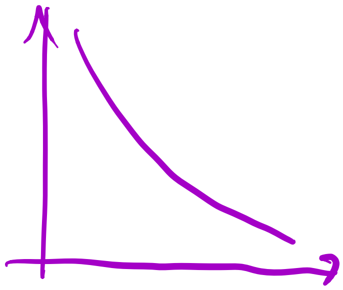




2. Decreasing convex

$f'(x) < 0$ $\forall x$

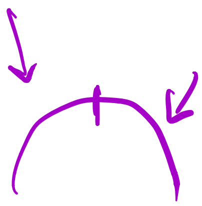
$f''(x) > 0$



3. Increasing concave

$f'(x) > 0$

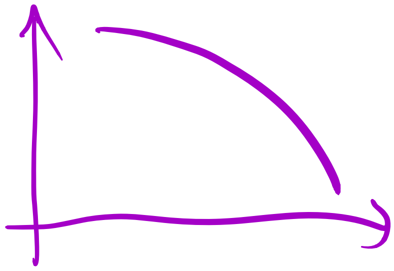
$f''(x) < 0$



4. Decreasing concave

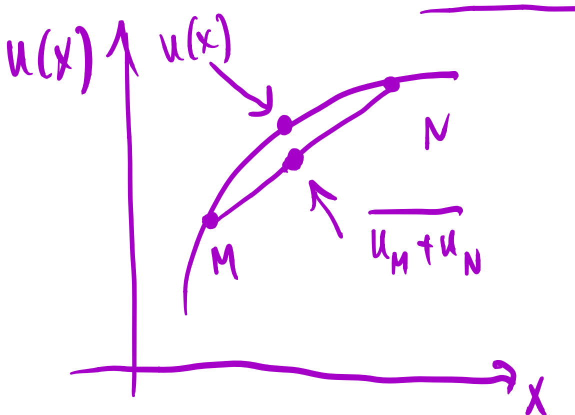
$f'(x) < 0$

$f''(x) < 0$

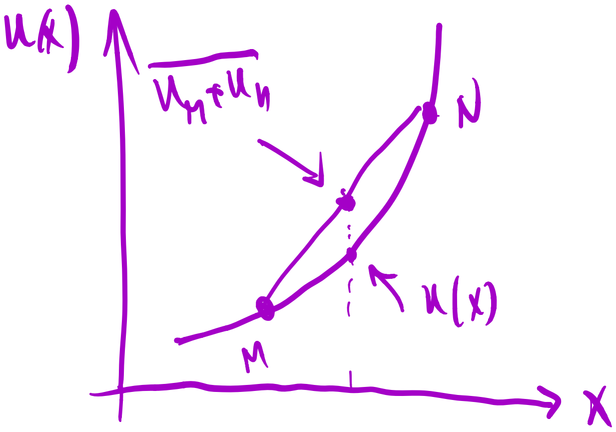


Example: Attitude towards risk.

✓Risk-averse preference U(x) is concave



✓Risk-loving preference U(x) is convex



Conclusion

Given a function, y = f(x), we can identify the maximum or minimum by finding

- FOC:

$\frac{dy}{dx} = 0$ Solve for critical value that its tangent line has slope = 0

- SOC:

$\frac{d^2y}{dx^2} > 0$ $f''(x_0) > 0$ minimum (slope increases)

$\frac{d^2y}{dx^2} < 0$ $f''(x_0) < 0$ maximum (slope decreases)

Example: Determine the extreme of $\boxed{f(x) = x^3 - 3x^2 + 2}$

FOC: $f'(x) = 3x^2 - 6x = 0$

$3x(x-2) = 0.$

$x^* = \{0, 2\}$

max.

SOC: $f''(x) = 6x - 6 \Rightarrow f''(0) = 6(0) - 6 = -6 < 0$

$f''(2) = 6(2) - 6 = 12 - 6 = 6 > 0$

min.

Example: $g(x) = \frac{x^4}{4} - \frac{3}{2}x^2$

$$g'(x) = \frac{\cancel{4}x^3}{\cancel{4}} - 3\cdot\frac{\cancel{2}x}{\cancel{2}} = \boxed{x^3 - 3x} = 0$$

$$x(x^2-3) = 0$$

$$x = \{0, \sqrt{3}, -\sqrt{3}\}$$

$$g''(x) = \frac{d}{dx}(x^3-3x) = 3x^2 - 3 \; : \; g''(0) = -3 < 0 \text{ Max}$$

$$g''(\sqrt{3}) = 6 > 0 \text{ Min}$$

$$g''(-\sqrt{3}) = 6 > 0 \text{ min}$$

$$\Rightarrow g(0) = \frac{0^4}{4} - \frac{3}{2}(0^2) = 0 \text{ max}$$

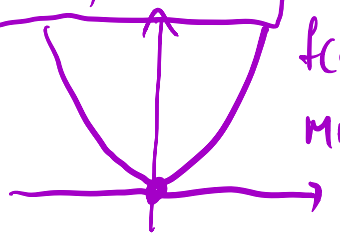
$$g(\sqrt{3}) = \frac{(\sqrt{3})^4}{4} - \frac{3}{2}(\sqrt{3})^2 = -\frac{9}{4}$$

$$g(-\sqrt{3}) = -\frac{9}{4} \quad \Big\} \text{ min}$$

Example: $f(x) = x^4$

FOC: $f'(x) = 4x^3 = 0 \Rightarrow \boxed{x = 0}$

SOC: $\boxed{f''(x) = 12x^2} \Rightarrow f''(0) = 0$



$f(x) = x^4$ minimum.

$\boxed{f''(x) = 12x^2} \Rightarrow \forall x, f''(x) \geq 0.$ ✓✓✓

Example: $f(x) = x^3$

FOC: $f'(x) = 3x^2 = 0 \Rightarrow x^* = 0$

SOC: $\boxed{f''(x) = 6x} \Rightarrow f''(0) = 6(0) = 0$ // $f''(x) = 6x$ → (+) if $x \geq 0$ ✓, (−) if $x < 0$ ✓

Therefore, when we get $f''(x_0) = 0$, it could be

1. Maximum or Minimum ex. $y = x^4$ slope changes its sign.
2. Inflection point ex. $y = x^3$ slope remains in the same sign.

Conditions	Maximum	Minimum
1. First-order necessary	$f'(x) = 0$	$f''(x) = 0$
✓ 2. Second-order necessary	$f''(x) \leq 0$	$f''(x) \geq 0$
✓ 3. Second-order sufficient	$f''(x) < 0$	$f''(x) > 0$

4 Profit maximization problem

Consider a profit function: Π (Q) = TR(Q) - TC(Q). A firm can optimize its objective function by choosing the amount of quantity produced so that it maximize the profit:

$$\max_{Q} \Pi(Q) = TR(Q) - TC(Q)$$

$\boxed{\text{FOC:}}$ Set $\frac{d\Pi(Q)}{dQ} = 0 \Rightarrow$

$$\begin{aligned}
\boxed{\frac{d}{dQ}\Pi(Q)} &\boxed{= 0} \\
\frac{d}{dQ}TR(Q) - \frac{d}{dQ}TC(Q) &= 0 \\
MR(Q) - MC(Q) &= 0 \\
\boxed{MR(Q) = MC(Q)} &\text{ profit-maximizing necessary condition}
\end{aligned}$$

Q^*

For max profit, we want $\boxed{\frac{d^2\Pi(Q)}{dQ^2} < 0}$

$\boxed{\text{SOC:}}$ $\frac{d^2\Pi(Q)}{dQ^2} < 0 \Rightarrow$

$$\begin{aligned}
\frac{d^2\Pi(Q)}{dQ^2} = \frac{d^2TR(Q)}{dQ^2} - \frac{d^2TC(Q)}{dQ^2} &< 0 \\
MR'(Q) - MC'(Q) &< 0 \\
\boxed{MR'(Q) < MC'(Q) \text{ sufficient condition}} \\
\text{slope of MR} &< \text{slope of MC}
\end{aligned}$$

$$\frac{d^2 TR(Q)}{dQ^2} = \overbrace{\frac{d}{dQ}\left(\overbrace{\frac{d}{dQ}TR(Q)}^{MR}\right)}^{MR'}$$

$$\frac{d^2 TC(Q)}{dQ^2} = \underbrace{\frac{d}{dQ}\left(\underbrace{\frac{d}{dQ}TC(Q)}_{MC}\right)}_{MC'}$$

Two cases to consider:
1. Perfect Competition
2. Monopoly

4.1 Perfect Competition

Assumptions:
1. many buyers and sellers⇒firms as price taker
2. homogeneous product
3. no barrier to entry or exit etc.

$$\mathrm{TR(Q)} = \bar{P} \cdot Q$$

$$\mathrm{TC(Q)} = \mathrm{C(Q)}$$

$$\Rightarrow \Pi(Q) = TR(Q) - TC(Q)$$

FOC: $\pi(Q) = \bar{P} \cdot Q - C(Q) \Rightarrow \pi' = \frac{d\pi}{dQ} = \bar{P} - \underbrace{C'(Q)}_{MC}$

$$\frac{d}{dQ}\Pi(Q) = \bar{P} - MC(Q) = 0$$

$$\boxed{\bar{P} = MC(Q)}$$

SOC:

$$\frac{d^2}{dQ^2}\Pi(Q) < 0 \quad \text{max.}$$

$$\frac{d}{dQ}\left[\bar{P} - MC(Q)\right] = 0 - MC'(Q) < 0$$

$$0 - MC'(Q) < 0$$

$$\boxed{MC'(Q) > 0}$$

Example: Given the following information, $\boxed{P = 10 \ , \ TC = Q^2}$. Find Π_{max}, Q and show that Π is maximum.

$$\pi = \underbrace{P \cdot Q}_{TR} - TC = 10 \cdot Q - Q^2$$

FOC : $\frac{d\pi}{dQ} = 10 - 2Q = 0 \Rightarrow Q^* = \frac{-10}{-2} = 5$

SOC : $\frac{d^2\pi}{dQ^2} = -2 < 0$ max.

Example: $\boxed{P = 30 \ , \ TC = 100 + 19Q - 5Q^2 + \frac{1}{3}Q^3}$. Find Q^* maximizing profit

$$\pi = P \cdot Q - TC = 30Q - 100 - 19Q + 5Q^2 - \frac{1}{3}Q^3$$

FOC : $\frac{d\pi}{dQ} = 30 - 19 + 10Q - Q^2 = 0$ $\boxed{11 + 10Q - Q^2 = 0}$

$Q^2 - 10Q - 11 = 0$

$(Q - 11)(Q + 1) = 0$

$Q^* = \cancel{-1}, 11$

SOC : $\frac{d^2\pi}{dQ^2} = \cancel{2Q - 10}$ not correct b/c we multiply (-1) to the slope equation

$$\frac{d^2\pi}{dQ^2} = \frac{d^2}{dQ^2}\left(11 + 10Q - Q^2\right) = 10 - 2Q$$

$\frac{d^2\pi(Q^*)}{dQ^2} = 10 - 2(11) = -12 < 0$

max.

4.2 Monopoly

Producer/seller can set the price in the market.

TR(Q) = P(Q) · Q

TC(Q) = C(Q)

PC : TR = $\bar{P} \cdot Q$

$\Rightarrow \Pi(Q) = TR(Q) - C(Q)$ $\Longrightarrow$ $\pi(Q) = P(Q) \cdot Q - C(Q)$

$\frac{d\pi(Q)}{dQ} = \left[P' \cdot Q + \frac{dQ}{dQ} \cdot P(Q)\right] - C'(Q)$

FOC:

$$\frac{d\Pi}{dQ} = [P(Q) \cdot (1) + Q \cdot P'(Q)] - MC(Q) = 0$$
$$MR(Q) = MC(Q)$$

SOC:

$$\frac{d^2}{dQ^2}\Pi(Q) < 0 \Rightarrow MR'(Q) < MC'(Q)$$

Example: P= 4000 - 33Q, TC = $2Q^3 - 3Q^2 + 400Q - 5000$, find Q^*_{max} that maximize Π.

$\pi = TR - TC = P \cdot Q - TC = (4000 - 33Q) \cdot Q - 2Q^3 + 3Q^2 - 400Q + 5000$

$\frac{d\pi}{dQ} = 4{,}000 - 66Q - 6Q^2 + 6Q - 400 = 0 \Rightarrow -6Q^2 - 60Q + 3600 = 0$

$\boxed{-Q^2 - 10Q + 600 = 0}$

$Q^2 + 10Q - 600 = 0$

$(Q - 20)(Q + 30) = 0 \Rightarrow Q^* = 20, \cancel{-30}$

$\frac{d^2\pi}{dQ^2} = -2Q - 10$

$\frac{d^2\pi}{dQ^2}(20) = -2(20) - 10 < 0$ max.

Example: $\boxed{P = 48 - 0.5Q}$, $TC = 2 + 60Q - 8Q^2 + Q^3$. Find Q^*

$$\Pi(Q) = \underbrace{(48 - 0.5Q)Q}_{\text{TR}} - \underbrace{2 - 60Q + 8Q^2 - Q^3}_{\text{TC}}$$

$Q^* = 1, 4$

$\frac{d^2\Pi}{dQ^2}(1) \gtrless 0$ min ✓

$\frac{d^2\Pi(4)}{dQ^2} \gtrless 0$ max. ✓

Monopoly power

$$\begin{aligned}
TR &= P(Q) \cdot Q \\
\frac{dTR}{dQ} &= \boxed{P(Q)\frac{dQ}{dQ} + Q\frac{dP(Q)}{dQ}} \\
MR &= P(Q) + Q\frac{d}{dQ}P(Q) \cdot \boxed{\frac{P(Q)}{P(Q)}} \\
&= P(Q) + \left[\frac{Q}{P(Q)} \cdot \frac{d}{dQ}P(Q)\right] P(Q) \\
MR &= P(Q) + \frac{1}{\varepsilon_d} P(Q) \\
\therefore MR &= P(Q)\left[1 + \frac{1}{\varepsilon_d}\right] \qquad \text{or} \qquad MR = P(Q)[1 - \frac{1}{|\varepsilon_d|}]
\end{aligned}$$

$Q \frac{dP(Q)}{dQ} \cdot \frac{P(Q)}{P(Q)}$

$\varepsilon_d < 0.$

$\% \Delta P \uparrow \Rightarrow \% \Delta Q \downarrow$

$$\begin{aligned}
\text{At profit-maximizing condition } MR(Q) &= MC(Q) \\
P(Q)\cdot[1-\tfrac{1}{|\varepsilon_d|}] &= MC(Q) \\
1-\tfrac{1}{|\varepsilon_d|} &= \tfrac{MC(Q)}{P(Q)} \\
&\vdots \\
\tfrac{MC(Q)-P(Q)}{P(Q)} &= \tfrac{-1}{|\varepsilon_d|} \\
\Leftrightarrow \boxed{\tfrac{P(Q)-MC(Q)}{P(Q)} = 1-\tfrac{1}{|\varepsilon_d|}}
\end{aligned}$$

$\Rightarrow$ Lerner's index

$\frac{P(Q)-MC(Q)}{P(Q)} = 1-\frac{1}{|\varepsilon_d|}$

$|\varepsilon_d| \uparrow \Rightarrow \frac{1}{|\varepsilon_d|} \downarrow \Rightarrow 1-\frac{1}{|\varepsilon_d|} \uparrow$

price elastic $\Rightarrow$ $P-MC \uparrow\uparrow$ margin

price inelastic

$|\varepsilon_d| \downarrow \Rightarrow \frac{1}{|\varepsilon_d|} \uparrow \Rightarrow 1-\frac{1}{|\varepsilon_d|} \downarrow \Rightarrow P-MC \downarrow$ margin

In PC, $|\varepsilon_d| = \infty$ because producers are price taker

$$\therefore \frac{P(Q)-MC(Q)}{P(Q)} = \frac{1}{\infty} = 0 \Rightarrow P = MC(Q)$$

In monopoly, $|\varepsilon_d| < \infty \Rightarrow \frac{1}{|\varepsilon_d|} > 0$

$$\therefore \frac{P(Q)-MC(Q)}{P(Q)} > 0 \Rightarrow P(Q) > MC(Q)$$

5 Effect of taxes

Consider again the profit function of a firm: $\Pi(Q) = TR(Q) - TC(Q)$, what will happen to Q^* if government imposes

1. lump-sum tax: T = t_0 fixed amount
2. profit tax : T = tΠ where 0 <t <1
3. specific tax : T = tQ where ~~0 <t <1~~

✓lump-sum tax

Π (Q) = $TR - TC - t_0 \Rightarrow \frac{d\Pi}{dQ} = MR - MC - 0 \Rightarrow \boxed{MR = MC}$

no impact on Q^*.

✓profit tax

Π (Q) = $\underbrace{TR - TC}_{\text{profit}} - \underbrace{t\Pi}_{\text{profit tax}} = TR - TC - t(TR - TC) = (1-t)[TR - TC]$

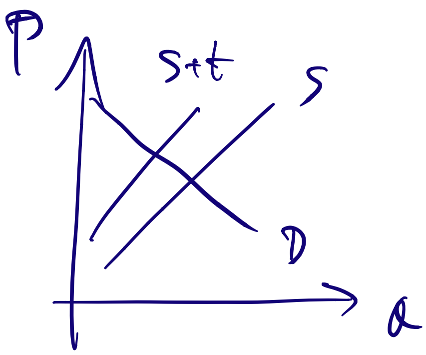
$\Rightarrow \frac{d\Pi}{dQ} = (1-t)(MR - MC) = 0$

$MR - MC = 0 \Rightarrow \boxed{MR = MC}$

no impact on Q^*.

✓specific tax

Π (Q) = $TR - TC - tQ \Rightarrow \frac{d\Pi}{dQ} = MR - MC - t = 0.$



$\boxed{MR = MC + t}$

Q^* will be lower.

6 Maximization of tax revenue

$\nearrow P \cdot Q$

Consider a firm facing a demand function $P = a - bQ$ which means that its total revenue is $TR = aQ - bQ^2$, given TC $= c_0 + c_1Q + c_2Q^2$

Suppose government imposes tax \$t per unit;

TC after tax: $TC_T = c_0 + c_1Q + c_2Q^2 + tQ$

Π after tax: $\Pi_T = TR - TC_T = aQ - bQ^2 - [c_0 + c_1Q + c_2Q^2 + tQ]$

$= aQ - bQ^2 - c_0 - c_1Q - c_2Q^2 - tQ.$

Profit-maximizing condition:

FOC:

$$\frac{d}{dQ}\Pi_T = \underbrace{a - 2bQ}_{MR} \underbrace{- c_1 - 2c_2Q}_{MC} \underbrace{- t}_{tax} = 0$$

$$a - 2bQ = c_1 + 2c_2Q + t$$

$$Q^* = \frac{a - c_1 - t}{2(b + c_2)} \leftarrow$$

SOC:

$$\frac{d^2\Pi_T}{dQ^2} = -2b - 2c_2 < 0 \quad \text{max. if } b > 0,\ c_2 > 0$$

$\therefore$ Total tax revenue: $t \cdot Q^* = t \cdot \left(\frac{a - c_1 - t}{2(b + c_2)}\right)$

Tax-revenue maximizing condition:

$$\boxed{t^* = \frac{a-c_1}{2}}$$

$$\boxed{\max_t T(t)} = \boxed{\frac{(a-c_1-t)t}{2(b+c_2)}}$$

FONC: $\frac{d}{dt}T(t) = \frac{d}{dt}\left(\frac{(a-c_1-t^2)t^2}{2(b+c_2)}\right) = \frac{a-c_1-2t}{2(b+c_2)} = 0.$

SOSC: $\frac{d^2}{dt}T(t) = \frac{-2}{2(b+c_2)} < 0$ max. if $b>0,\ c_2>0$

Example: $\boxed{\text{P =40-0.5Q}}$, TC $= 2 - 5Q + 7Q^2$. Find t (specific tax) that maximizes total tax revenue.

1st Q^*, check Q^* max/min
2nd write tax revenue : $t \cdot Q^*$
3rd optimize obj. function.

① $\pi = 40Q - 0.5Q^2 - 2 + 5Q - 7Q^2 - tQ \Rightarrow \frac{d\pi}{dQ} \Rightarrow Q^* = \frac{45-t}{15}$

$\frac{d^2\pi}{dQ^2} = -15 < 0$ max.

② $T(t) = t \cdot \frac{45-t}{15}$

③ FOC : $\frac{dT(t)}{dt} = 0 \Rightarrow t^* = 22.5$

SOC : $\frac{d^2T}{dt^2} = -\frac{2}{15} < 0$ max.

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