CHAPTER 8

Optimization without Constraint: More-Than-One Independent Variable Cases

- Conditions for maximum or minimum
- Third degree price discrimination
- Competitive Firm Input Choices: Cobb-Douglas Technology
- Multiproduct-firm
- Multiplant-firm

Optimization Condition

One choice variable

The objective function: $z = f(x)$

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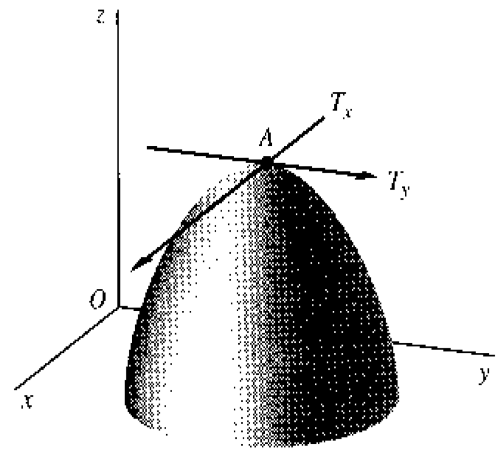
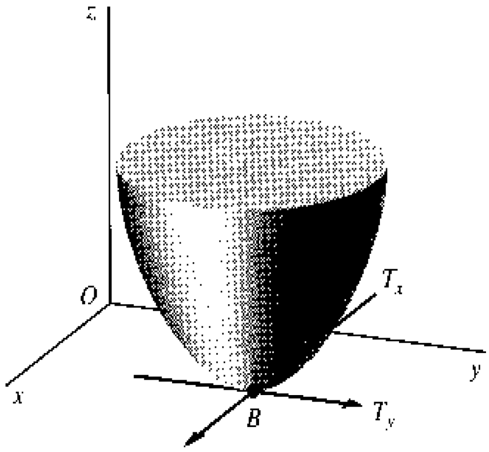
Condition	Maximum	Minimum
First-order necessary (FONC, FOC)	Derivative Total differential	Derivative Total differential
Second-order sufficient (SOSC, SOC)	Derivative Total differential	Derivative Total differential

Extreme Value of a Function of Two Variables

$$z = f(x, y)$$





<u>First-Order Necessary Condition</u>

$dz = 0$ for every dx and dy

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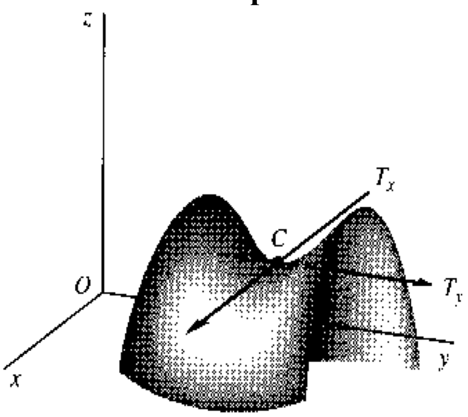
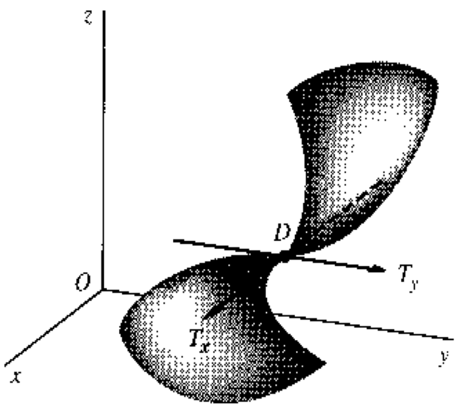
The first order necessary condition is necessary, but not sufficient to establish a maximum or a minimum.

Saddle point



Inflection point



Second order sufficient condition

The satisfaction of the first-order condition earmarks certain values of z as the stationary values of the objective function. If at a stationary value of z we find that d^2z is positive definite, i.e., $d^2z > 0$, this will suffice to establish that value of z as a minimum. Analogously, the negative definiteness of d^2z, $d^2z < 0$, is a sufficient condition for the stationary value to be a maximum.

Second-order total differential, Hessian Matrix and Definiteness of Hessian Matrix

🕮 **About "H_k"**

Definition : Let A be an $n \times n$ matrix. A $k \times k$ submatrix, deriving from deleting the last $n - k$ columns and the last $n - k$ rows of matrix A , can be called the k^{th} order leading principal submatrix of matrix A. The corresponding determinant of this $k \times k$ submatrix is called the k^{th} order leading principal minor of matrix A.

The k^{th} order leading principal submatrix of matrix A is denoted by A_k.
The k^{th} order leading principal minor of matrix A is denoted by $|A_k|$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Leading principal submatrices are:

$[a_{11}]$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Leading principal minors are:

$$|a_{11}|$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

For Hessian Matrix of function $y = f(x_1, x_2, \dots, x_n)$

$$H = \begin{bmatrix} f_{11} & f_{12} & f_{13} & \cdots & f_{1n} \\ f_{21} & f_{22} & f_{23} & \cdots & f_{2n} \\ f_{31} & f_{32} & f_{33} & \cdots & f_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & f_{n3} & \cdots & f_{nn} \end{bmatrix}$$

$$\_\_\_\_ = [f_{11}]$$

$$\_\_\_\_ = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

$$\_\_\_\_ = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

Leading principal minors are:

$$\_\_\_\_ = |f_{11}|$$

$$\_\_\_\_ = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$$

$$\_\_\_\_ = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

🕮 About "Definiteness of *H*"

Let ***H*** be a symmetric matrix with dimension $n \times n$

(a) ***H*** is **positive definite** if all n leading principal minors are positive

$$|H_1| > 0, |H_2| > 0, |H_3| > 0, \ldots$$

(b) ***H* is negative definite if** n leading principal minor duly alternate in sign with the first one being negative.

$$|H_1| < 0, |H_2| > 0, |H_3| < 0, \ldots$$

If (a) or (b) is not met, H is indefinite.

Example: Find definiteness of matrix $B = \begin{bmatrix} -3 & 4 \\ 4 & -6 \end{bmatrix}$

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Homework: Find definiteness of the following symmetric matrices ต่อไปนี้

(a) $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} -3 & 4 \\ 4 & -5 \end{bmatrix}$

(c) $\begin{bmatrix} -3 & 4 \\ 4 & -6 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{bmatrix}$

(f) $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{bmatrix}$

Second-Order Sufficient Condition (SOSC, SOC)

For the maximum of objective function $z = f(x, y)$

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For the minimum of objective function $z = f(x, y)$

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Example: Find the maximum/minimum of function $z = x + 2ey - e^x - e^{2y}$

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Objective functions with more three choice variables

$$z = f(x_1, x_2, x_3)$$

F.O.N.C :
$dz = 0$

$dz = f_1 dx_1 + f_2 dx_2 + f_3 dx_3 = 0; dx_1, dx_2, dx_3 > 0$

$f_1 = f_2 = f_3 = 0$

We can use $f_1 = f_2 = f_3 = 0$ to find the stationary values/critical values

S.O.S.C

$$H = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

All leading principal minor
$|H_1| = |f_{11}|$

$$|H_2| = \begin{vmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{vmatrix}$$

$$|H_3| = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix}$$

$$z \text{ is at the } \begin{bmatrix} \text{maximum} \\ \text{minimum} \end{bmatrix} \text{ if } \begin{cases} d^2z < 0 \\ d^2z > 0 \end{cases} \text{ if } \begin{cases} |H_1| < 0, |H_2| > 0, |H_3| < 0 \\ |H_1| > 0, |H_2| > 0, |H_3| > 0 \end{cases} \text{ if } \begin{cases} H \text{ is negative definite} \\ H \text{ is positive definite} \end{cases}$$

Objective functions with more n −choice variables

$$z = f(x_1, x_2, x_3, \dots, x_n)$$

F.O.N.C :

$$dz = f_1 dx_1 + f_2 dx_2 + \cdots + f_n dx_n = 0$$

$$\frac{\partial^2 f_i}{\partial x_i} = 0, for\ all\ i, i = 1,2, \dots n$$

S.O.S.C:

$$\therefore \ |H| = \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{vmatrix}$$

All principal minors $|H_1|$, $|H_2|$,… $|H_n|$

If $z\ is\ at$ is the maximum when $d^2z < 0$, H is negative definite:

$$|H_1| < 0, |H_2| > 0, |H_3| < 0, |H_4| > 0, |H_5| < 0, \dots$$

If $z\ is\ at$ is the minimum when $d^2z > 0$, H is positive definite:

$$|H_1| > 0, |H_2| > 0, |H_3| > 0, |H_4| > 0, |H_5| > 0, \dots$$

Homework:Find extreme values of the following functions and check for the sufficient condition

$z = 8x^3 + 2xy - 3x^2 + y^2 + 1$

$z = 2x_1^2 + x_1x_2 + 4x_2^2 + x_1x_3 + x_3^2 + 2$

$z = -3x_1^3 + 3x_1x_3 + 2x_2 - x_2^2 - 3x_3^2$

$z = x_1^2 + 3x_2^2 - 3x_1x_2 + 4x_2x_3 + 6x_3^2$

$z = x_1x_3 + x_1^2 - x_2 + x_2x_3 + x_2^2 + 3x_3^2$

$z = e^x + e^y + e^{w^2} - 2e^w - (x + y)$

$z = x^4 + x^2 - 6xy + 3y^2$

$w = x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$

$w = (x^2 + 2y^2 + 3z^2)e^{-(x^2+y^2+z^2)}$

Summary of optimal conditions

The objective function: $z = f(x, y)$

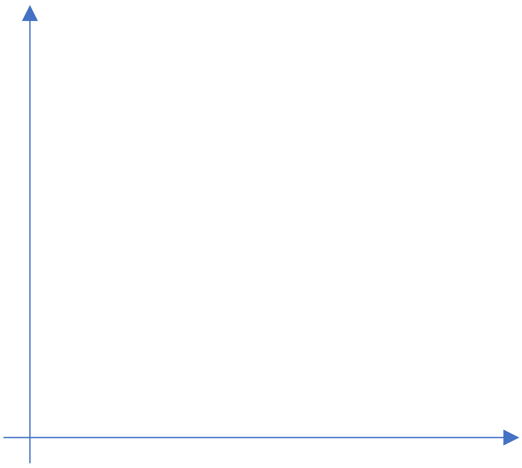
Condition	Maximum	Minimum
First-order necessary (FONC, FOC)	$f_x = f_y = 0$	$f_x = f_y = 0$
Second-order sufficient (SOSC, SOC)	$\lvert H_1 \rvert = f_{xx} < 0$ $\lvert H_2 \rvert = f_{xx}f_{yy} - f_{xy}^2 > 0$	$\lvert H_1 \rvert = f_{xx} > 0$ $\lvert H_2 \rvert = f_{xx}f_{yy} - f_{xy}^2 > 0$

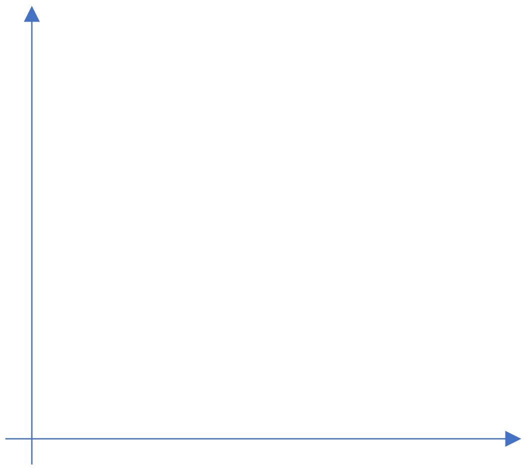
The objective function: $z = f(x_1, x_2, \dots, x_n)$

Condition	Maximum	Minimum
First-order necessary (FONC, FOC)	$f_1 = f_2 = \cdots = f_n = 0$	$f_1 = f_2 = \cdots = f_n = 0$
Second-order sufficient (SOSC, SOC)	H is negative definite. $(-1)^k \lvert H_k \rvert > 0$ $k = 1,2, \dots, n$	H is positive definite. $\lvert H_k \rvert > 0$ $k = 1,2, \dots, n$

Third- Degree Price Discrimination

Third-Degree Price Discrimination involves charging different prices to different segments of the market for the same good. This is to maximize profit within each segment of the market. For example, a theater may divide cinema goers into seniors, adults, and children, each paying a different price when seeing the same movie.





Total Revenue of a company is:

$$TR = R_1(Q_1) + R_2(Q_2) + R_3(Q_3),$$

R_i is a total revenue function of market i.

Total cost of the company is:

$$C = C(Q)$$

$$Q = Q_1 + Q_2 + Q_3$$

Q_i is quantity of product sold in market i.

Find the quantity and price in each market i that maximizes firm's profit.

<u>Step 1</u> State the objective Function: Profit function

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Step 2 Find FONC and use FONC to find critical values

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$$MC = MR_1 = MR_2 = MR_3$$

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Conclusion: In the market with lower price elasticity of demand (lower $|\varepsilon_{di}|$), the maximizing-profit price in that market will be...........

Step 3 Check SOSC whether $Q_1^*, P_1^*, Q_2^*, P_2^*, Q_3^*, P_3^*$ indeed give the maximum profit

Hessian Matrix:

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{bmatrix}$$

$$H = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Check all leading principal minors

$|H_1| = |\pi_{11}|$

$|H_2| = \begin{vmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{vmatrix}$

$|H_3| = \begin{vmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{vmatrix}$

Example: Suppose a monopolist firm sells its product in three markets with corresponding market demand:

Domestic market: $P_1 = 63 - 4Q_1$
Asia market: $P_2 = 105 - 5Q_2$
Europe market: $P_3 = 75 - 6Q_3$

Total cost function of this firm is:

$$C = 20 + 15Q;$$
$$Q = Q_1 + Q_2 + Q_3$$

Find total quantity produced by the monopolist and price and quantity in each market that maximize profit.

<u>Step 1</u> State the objective Function: Profit function

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<u>Step 2</u> Find FONC and use FONC to find critical values

<u>Step 3</u>: SOSC

$$H = \begin{bmatrix} \quad \end{bmatrix}$$

Check all leading principal minors

$|H_1| =$

$|H_2| =$

$|H_3| =$

Competitive Firm Input Choices: Cobb-Douglas Technology

Let a firm in competitive market have Cobb-Douglas Production Function:

$$Q = L^{\alpha} K^{\beta}$$

Let wage be w, rent be r baht per unit.

Characteristics of the production function:

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<u>Step 1</u> State the objective Function: Profit function

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<u>Step 2</u> Find FONC and use FONC to find critical values

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The Economic interpretation

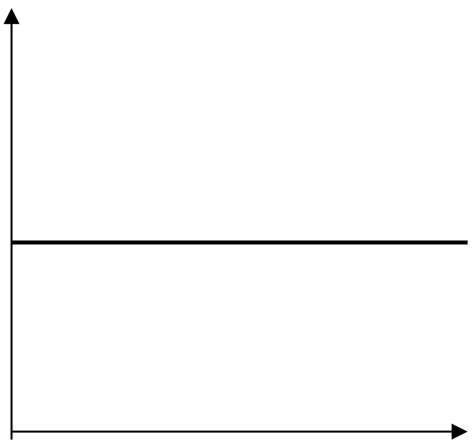
<u>Step 3</u>: SOSC

$$H = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Check all leading principal minors

$|H_1| =$

$|H_2| =$



Multiproduct Firm

Multiproduct Firm in competitive market

Let a firm produce and sell two products. Each product is sold in competitive market, hence the firm is price taker. Total Revenue is:

$TR =$

P_1 is market price of good 1.
P_2 is market price of good 2.
Q_1 is quantity of good 1 sold in the market.
Q_2 is quantity of good 2 sold in the market.

Total cost of this firm is:

$$C = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$$

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Firm's Profit Function

$$\pi = TR - TC$$

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Choice Variables: …………………………………………………

Find FONC and use FONC to find critical values

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Check Second Order Sufficient Condition (SOSC)

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

$$H = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$|H_1| =$

$|H_2| =$

The second order differential of π, $d^2\pi$, is ....definite. Hence, the profit is………………..

Multiproduct Firm in monopoly market

Let market demand of good 1 is $Q_1 = 40 - 2P_1 + P_2$
Let market demand of good 2 is $Q_2 = 15 + P_1 - P_2$

From market demand, good 1 and good 2 are ....

The market demands are given as functions of P_1, P_2. Since we would like to choose Q_1, Q_2, we need to convert the above market demands to be in the form of $P_1 = f(Q_1, Q_2)$, and $P_2 = g(Q_1, Q_2)$.

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Then, we can use the inverse market demand functions to get total revenue function:

$TR =$……………………………………………………………………………………………

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Let the cost function of the monopolist be

$$C = 2Q_1^2 + Q_1Q_2 + 2Q_2^2$$

Profit Function:

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Choice Variables: …………………………………………………

Find FONC and use FONC to find critical values

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Check Second Order Sufficient Condition (SOSC)

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

$$H = \begin{bmatrix} & \\ & \\ & \end{bmatrix}$$

$|H_1| =$

$|H_2| =$

The second order differential of π, $d^2\pi$, is ....definite. Hence, the profit is………………..

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Multiplant Firm Problem

Consider a firm with many plants. Each plant i has total cost TC_i.

$$TC_i = C_i(q_i),\ i = 1,2,\dots,n$$

q_i is the level of output produced by factory i.

Assume that this firm sells its product in one market. Firm's total revenue function is:

$TR =$……………………………………………………………………………………

$Q =$…………………………………

If output market is competitive market,

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If output market is monopoly market,

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Firm's Profit Function

$$\pi = TR - TC$$

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Choice Variables: …………………………………………………

NOTE: Third-degree price discrimination vs. Multiplant firm

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First Order Necessary Condition:

π_i = ………………………………………………………………for all $i = 1,2, \dots, n$

SOSC:

$$H = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$|H_1| = R'' - C_1'' < 0$

$|H_2| =$

Economic interpretation:

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Example: Consider a monopolist firm with two plants:

Factory 1: $C_1(Q_1) = 10Q_1^2$

Factory 1: $C_2(Q_2) = 20Q_2^2$

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Firm's market demand is $P = 700 - 5Q$, $Q = Q_1 + Q_2$

Draw MC_1, MC_2, MC_T, AR, MR and indicate the maximizing level of output from each plant and the profit-maximizing price.

Find Q_1, Q_2, Q_{Total}, P that maximize profit of this firm.

Profit Function:

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Choice Variables: …………………………………………………

Find FONC and use FONC to find critical values

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Check Second Order Sufficient Condition (SOSC)

$$H = \begin{bmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{bmatrix}$$

$$H = \begin{bmatrix} & \\ & \\ & \end{bmatrix}$$

$|H_1| =$

$|H_2| =$

The second order differential of π, $d^2\pi$, is ....definite. Hence, the profit is………………..

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Summary:

3rd price discrimination:

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Choosing levels of factor of production:

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Multiproduct:

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Multiplant:

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