Exericse Solution: Solving Inequality (Part III)

1. Let a_1, a_2, a_3, and a_4 be some constant real numbers such that

$$a_1 < a_2 < a_3 < a_4.$$

Determine the solution set for each of the following inequalities in terms of a_1, a_2, a_3, and a_4.

(a) $(x - a_1)(x - a_2)(x - a_3)(x - a_4) \geq 0$

(b) $(x - a_1)^2(x - a_2)(x - a_3)^2(x - a_4)^3 \geq 0$

Solution To solve $(x-a_1)(x-a_2)(x-a_3)(x-a_4) \geq 0$, since $a_1 < a_2 < a_3 < a_4$, we will consider sub-intervals divided using the points $x = a_1, a_2, a_3, a_4$ as shown in the table.

	$x \in (-\infty, a_1)$	$x \in (a_1, a_2)$	$x \in (a_2, a_3)$	$x \in (a_3, a_4)$	$x \in (a_4, \infty)$
$x - a_1$	$-$	$+$	$+$	$+$	$+$
$x - a_2$	$-$	$-$	$+$	$+$	$+$
$x - a_3$	$-$	$-$	$-$	$+$	$+$
$x - a_4$	$-$	$-$	$-$	$-$	$+$
$(x - a_1)(x - a_2)(x - a_3)(x - a_4)$	$\boxed{+}$	$-$	$\boxed{+}$	$-$	$\boxed{+}$

That is, the solution set is given by $(-\infty, a_1] \cup [a_2, a_3] \cup [a_4, \infty)$. ■

(b)

To solve $(x-a_1)^2(x-a_2)(x-a_3)^2(x-a_4)^3 \geq 0$, since $a_1 < a_2 < a_3 < a_4$ and $(x-a_1)^2, (x-a_3)^2 \geq 0$, we will consider $(x - a_2)(x - a_4)^3 \geq 0$ and use sub-intervals divided by the points $x = a_2, a_4$ together with the set $\{a_1, a_2, a_3, a_4\}$ as shown in the table.

	$x \in (-\infty, a_2)$	$x \in (a_2, a_4)$	$x \in (a_4, \infty)$
$x - a_2$	$-$	$+$	$+$
$x - a_4$	$-$	$-$	$+$
$(x - a_2)(x - a_4)^3$	$\boxed{+}$	$-$	$\boxed{+}$

That is, the solution set is given by $(-\infty, a_2] \cup [a_4, \infty) \cup \{a_3\}$.

Note that $a_1 \in (-\infty, a_2]$.

■

2. Let a and b be real numbers with $|b| < |a| < 1$ and $a > 0$.

(a) Show that $a - 1 < 0 < b + a < 2a < 1 + a$.

(b) (Modified) Show that

$$\frac{2a^2 - 2a}{a + 1} > \frac{a^2 - 1}{a + b}.$$

Remark: The original version $\frac{2a^2-2a}{b+a} > \frac{a^2-1}{2a}$ is incorrect and this problem will not be graded.

Solution:

(a) $|a| < 1$ implies

$$-1 < a < 1 \quad \Rightarrow \quad -a > -1 \tag{1}$$

and $|b| < |a|$ and $a > 0$ imply

$$-|a| < b < |a| \quad \text{and} \quad -a < b < a \quad \text{since} \quad |a| = a. \tag{2}$$

From the inequalities in (1) and (2), we have

$$-1 < -a < b < a < 1$$

and adding a throughout the above inequalities gives

$$a - 1 < 0 < b + a < 2a < 1 + a.$$

(b) First notice the given inequality in (b) can be rewritten as

$$\frac{2a(a-1)}{b+a} > \frac{(a-1)(a+1)}{2a}.$$

From (a),

$$\begin{aligned} & 0 < 2a < a + 1 \\ 0 < b + a < a + 1 \Rightarrow \quad & 0 < \tfrac{1}{a+1} < \tfrac{1}{a+b} \end{aligned}$$

Multiply the above two inequalities gives,

$$0 < \frac{2a}{a+1} < \frac{a+1}{a+b}$$

and since $a - 1 < 0$,

$$0 > \frac{2a(a-1)}{a+1} > \frac{(a+1)(a-1)}{a+b} \quad \text{or} \quad \frac{2a^2 - 2a}{a+1} > \frac{a^2 - 1}{a+b}.$$

■

3. Find the solution set for each of following inequalities.

(a)

$$\frac{1}{x} < 8$$

(b)

$$\frac{9-x}{x-2} \geq \frac{2-x}{x-9}$$

(c)

$$\frac{|2x+5| - 2}{3|x| - 1} \leq 1$$

(d)

$$\frac{|x^2 - x + 1|}{|3x + 1|} > 1$$

(e)

$$\frac{|x^2 - 2| + 2}{x^2 - 3|x| + 2} < 0$$

Solution:

(a)

$$\frac{1}{x} < 8$$

Solution: $(-\infty, 0) \cup (1/8, \infty)$

$$\begin{aligned} \frac{1}{x} &< 8 \\ \frac{1}{x} - 8 &< 0 \\ \frac{1 - 8x}{x} &< 0 \\ \frac{x - 1/8}{x} &> 0 \end{aligned}$$

We consider the intervals divided by $x = 0, 1/8$ as follows.

	$x \in (-\infty, 0)$	$x \in (0, 1/8)$	$x \in (1/8, \infty)$
$\frac{x-1/8}{x}$	$\frac{(-)}{(-)}$	$\frac{(+)}{(-)}$	$\frac{(+)}{(+)}$
	$\boxed{+}$	$-$	$\boxed{+}$

Hence, from the table, the solution set is $(-\infty, 0) \cup (1/8, \infty)$.

■

(b)

$$\frac{9 - x}{x - 2} \geq \frac{2 - x}{x - 9}$$

Solution: $(2, 11/2] \cup (9, \infty)$

$$\begin{aligned}
\frac{9-x}{x-2} &\geq \frac{2-x}{x-9} \\
\frac{9-x}{x-2} - \frac{2-x}{x-9} &\geq 0 \\
\frac{-(x-9)^2+(x-2)^2}{(x-2)(x-9)} = \frac{(x-2)^2-(x-9)^2}{(x-2)(x-9)} &\geq 0 \\
\frac{(x-2-x+9)(x-2+x-9)}{(x-2)(x-9)} &\geq 0 \\
\frac{7(2x-11)}{(x-2)(x-9)} &\geq 0
\end{aligned}$$

We consider the intervals divided by $x = 2, 11/2, 9$ as follows.

	$x \in (-\infty, 2)$	$x \in (2, 11/2)$	$x \in (11/2, 9)$	$x \in (9, \infty)$
$\frac{7(2x-11)}{(x-2)(x-9)}$	$\frac{(-)}{(-)(-)} = (-)$	$\frac{(-)}{(+)(-)} = \boxed{(+)}$	$\frac{(+)}{(+)(-)} = (-)$	$\frac{(+)}{(+)(+)} = \boxed{(+)}$

Note that we consider $\geq$, so $\{11/2\}$ has to be included in the solution set, but not the points that give zero denominator. Hence, from the table, the solution set is $(2, 11/2] \cup (9, \infty)$.

■

(c)

$$\frac{|2x+5|-2}{3|x|-1} \leq 1$$

By using definition of absolute value, we have $|2x+5| = \begin{cases} -(2x+5), & 2x+5<0 \Leftrightarrow x < -5/2 \\ 2x+5, & 2x+5 \geq 0 \Leftrightarrow x \geq -5/2 \end{cases}$,

and $|x| = \begin{cases} -x, & x<0 \\ x, & x \geq 0 \end{cases}$, we will consider 3 cases:

Case I: $x \in (-\infty, -5/2)$, Case II:$x \in [-5/2, 0)$, Case III: $x \in [0, \infty)$.

Case I: $x \in (-\infty, -5/2)$.

$$\begin{aligned}
\frac{|2x+5|-2}{3|x|-1} &\leq 1 \\
\frac{-(2x+5)-2}{-3x-1} &\leq 1 \quad \Leftrightarrow \quad \frac{2x+7}{3x+1} \leq 1 \quad \Leftrightarrow \quad \frac{2x+7}{3x+1} - 1 \leq 0 \quad \Leftrightarrow \quad \frac{(2x+7)-(3x+1)}{3x+1} \leq 0 \\
\frac{-x+6}{3x+1} &\leq 0 \quad \text{or} \quad \frac{x-6}{3x+1} \geq 0
\end{aligned}$$

We consider the intervals divided by $x = -1/3, 6$ as follows.

	$x \in (-\infty, -1/3)$	$x \in (-1/3, 6)$	$x \in (6, \infty)$
$\frac{x-6}{3x+1}$	$\frac{(-)}{(-)} = \boxed{(+)}$	$\frac{(-)}{(+)} = (-)$	$\frac{(+)}{(+)} = \boxed{(+)}$

Note that we consider $\geq$, so $\{6\}$ has to be included in the solution set, but not the points that give zero denominator. Hence, from the table, $x \in (-\infty, -1/3) \cup [6, \infty)$. That is, in this case, the solution set is $(-\infty, -5/2) \cap \{(-\infty, -1/3) \cup [6, \infty)\} = (-\infty, -5/2)$.

Case II:$x \in [-5/2, 0)$,

$$\begin{aligned}
\frac{|2x+5|-2}{3|x|-1} &\leq 1 \\
\frac{(2x+5)-2}{-3x-1} &\leq 1 \quad \Leftrightarrow \quad \frac{2x+3}{3x+1} \geq -1 \quad \Leftrightarrow \quad \frac{2x+3}{3x+1}+1 \geq 0 \quad \Leftrightarrow \quad \frac{(2x+3)+(3x+1)}{3x+1} \geq 0 \\
\frac{5x+4}{3x+1} &\geq 0
\end{aligned}$$

We consider the intervals divided by $x = -4/5, -1/3$ as follows.

	$x \in (-\infty, -4/5)$	$x \in (-4/5, -1/3)$	$x \in (-1/3, \infty)$
$\frac{5x+4}{3x+1}$	$\frac{(-)}{(-)} = \boxed{(+)}$	$\frac{(-)}{(+)} = (-)$	$\frac{(+)}{(+)} = \boxed{(+)}$

Note that we consider $\leq$, so $\{-4/5\}$ has to be included in the solution set, but not the points that give zero denominator. Hence, from the table, $(-1/3, 2]$. That is, in this case, the solution set is $[-5/2, 0) \cap \{(-\infty, -4/5] \cup (-1/3, \infty)\} = [-5/2, -4/5] \cup (-1/3, 0)$.

Case III: $x \in [0, \infty)$.

$$\begin{aligned}
\frac{|2x+5|-2}{3|x|-1} &\leq 1 \\
\frac{(2x+5)-2}{3x-1} &\leq 1 \quad \Leftrightarrow \quad \frac{2x+3}{3x-1} \leq 1 \quad \Leftrightarrow \quad \frac{2x+3}{3x-1}-1 \leq 0 \quad \Leftrightarrow \quad \frac{(2x+3)-(3x-1)}{3x-1} \leq 0 \\
\frac{-x+4}{3x-1} &\leq 0 \quad \text{or} \quad \frac{x-4}{3x-1} \geq 0
\end{aligned}$$

We consider the intervals divided by $x = 1/3, 4$ as follows.

	$x \in (-\infty, 1/3)$	$x \in (1/3, 4)$	$x \in (4, \infty)$
$\frac{x-4}{3x-1}$	$\frac{(-)}{(-)} = \boxed{(+)}$	$\frac{(-)}{(+)} = (-)$	$\frac{(+)}{(+)} = \boxed{(+)}$

Note that we consider $\geq$, so $\{4\}$ has to be included in the solution set, but not the points that give zero denominator. Hence, from the table, $(-\infty, 1/3) \cup [4, \infty)$. That is, in this case, the solution set is $[0, \infty) \cap \{(-\infty, 1/3) \cup [4, \infty)\} = [0, 1/3) \cup [4, \infty)$.

From cases (I)-(III), we have that the solution set is given by $(-\infty, -5/2) \cup \{[-5/2, -4/5] \cup (-1/3, 0)\} \cup \{[0, 1/3) \cup [4, \infty)\} = (-\infty, -\frac{4}{5}) \cup (-\frac{1}{3}, \frac{1}{3}) \cup [4, \infty)$. ■

(d)

$$\frac{|x^2-x+1|}{|3x+1|} > 1$$

Solution: $(-\infty, -1/3) \cup (-1/3, 0) \cup (4, \infty)$

There are many approaches for solving this problem. Two approaches are considered here.

Approach 1:

Since both sides are positive, the above inequality is equivalent to

$$\left(\frac{|x^2-x+1|}{|3x+1|}\right)^2 > 1^2 \quad \Leftrightarrow \quad \left(\frac{x^2-x+1}{3x+1}\right)^2 - 1^2 > 0 \quad \Leftrightarrow$$

$$\begin{aligned} \left(\frac{x^2-x+1}{3x+1} - 1\right)\left(\frac{x^2-x+1}{3x+1} + 1\right) &> 0 \\ \left(\frac{x^2-x+1-3x-1}{3x+1}\right)\left(\frac{x^2-x+1+3x+1}{3x+1}\right) &> 0 \\ \frac{(x^2-4x)(x^2+2x+2)}{(3x+1)^2} &> 0. \end{aligned}$$

Consider x^2+2x+2. Since $b^2-4ac = 4-4(1)(2) < 0$ and $a = 1 > 0$, then $x^2+2x+2 > 0 \quad \forall x \in \mathbb{R}$ and the inequality becomes:

$$\begin{aligned} x^2-2x+2 > 0 \quad \forall x \in \mathbb{R} \quad \Rightarrow \quad \frac{x(x-4)}{(3x+1)^2} &> 0 \\ (3x+1)^2 \geq 0 \quad \forall x \in \mathbb{R}, \quad \text{for } x \neq -1/3, \quad \Rightarrow \quad x(x-4) &> 0 \end{aligned}$$

By setting $x(x-4) = 0$, we consider the intervals divided by $x = 0, 4$ as follows.

	$x \in (-\infty, 0)$	$x \in (0, 4)$	$x \in (4, \infty)$
$x(x-4)$	$(-)(-) = \boxed{(+)}$	$(+)(-) = (-)$	$(+)(+) = \boxed{(+)}$

Hence, from the table and from the fact that $x \neq -1/3$, $\{(-\infty, 0) \cup (4, \infty)\} - \{-1/3\} = (-\infty, -1/3) \cup (-1/3, 0) \cup (4, \infty)$.

■

Approach 2:

Consider x^2-x+1. Since $b^2-4ac = 1-4(1)(1) < 0$ and $a = 1 > 0$, then $x^2-x+1 > 0 \quad \forall x \in \mathbb{R}$ and $|x^2-x+1| = x^2-x+1$. So the inequality becomes:

$$\frac{x^2-x+1}{|3x+1|} > 1$$

and we only need to consider two cases: $|3x+1| = \begin{cases} -(3x+1), & 3x+1 < 0 \ \Leftrightarrow x < -1/3 \\ 3x+1, & 3x+1 \geq 0 \ \Leftrightarrow x \geq -1/3 \end{cases}$,

(*) Note that, since $|3x+1|$ is the denominator, we must have $x \neq -1/3$ and we consider (I) $x < -1/3$ and (II) $x > -1/3$.

(I) For $x < -1/3$, $|3x+1| = -(3x+1)$

$$\frac{x^2-x+1}{|3x+1|} > 1 \quad \Leftrightarrow \quad x^2-x+1 > |3x+1| \quad \Leftrightarrow \quad x^2-x+1 > -3x-1 \quad \Leftrightarrow \quad x^2+2x+2 > 0$$

which is true for all real numbers x (consider x^2+2x+2. Since $b^2-4ac = 4-4(1)(2) < 0$ and $a = 1 > 0$, then $x^2+2x+2 > 0 \quad \forall x \in \mathbb{R}$). Hence, the solution set in this case is $(-\infty, -1/3) \cap \mathbb{R} = (-\infty, -1/3)$.

(II) For $x > -1/3$, $|3x+1| = 3x+1$

$$\frac{x^2-x+1}{|3x+1|} > 1 \quad\Leftrightarrow\quad x^2-x+1 > |3x+1| \quad\Leftrightarrow\quad x^2-x+1 > 3x+1 \quad\Leftrightarrow\quad x^2-4x > 0.$$

By setting $x(x-4) = 0$, we consider the intervals divided by $x = 0, 4$ as follows.

	$x \in (-\infty, 0)$	$x \in (0, 4)$	$x \in (4, \infty)$
$x(x-4)$	$(-)(-) = (+)$	$(+)(-) = (-)$	$(+)(+) = (+)$

Hence, from the table and from the fact that $x > -1/3$, $(-1/3, \infty) \cap \{(-\infty, 0) \cup (4, \infty)\} = (-1/3, 0) \cup (4, \infty)$.

From (I) and (II), the solution set is $(-\infty, -1/3) \cup (-1/3, 0) \cup (4, \infty)$. ■

(e)

$$\frac{|x^2-2|+2}{x^2-3|x|+2} < 0$$

Solution: $(-2, -1) \cup (1, 2)$

Notice that $|x^2-2| \geq 0 \Rightarrow |x^2-2|+2 \geq 2 > 0$ for any real number x. That is, to find x that gives $\frac{|x^2-2|+2}{x^2-3|x|+2} < 0$, we can solve

$$\begin{aligned} \frac{1}{x^2-3|x|+2} &< 0 \\ \frac{1}{|x|^2-3|x|+2} &< 0 \\ \frac{1}{(|x|-1)(|x|-2)} &< 0. \end{aligned}$$

By setting $(|x|-1)(|x|-2) = 0$, we consider the intervals divided by $|x| = 1, 2$ together with the fact that $|x| \geq 0$ as shown below.

	$\lvert x\rvert \in [0, 1)$	$\lvert x\rvert \in (1, 2)$	$\lvert x\rvert \in (2, \infty)$
$(\lvert x\rvert-1)(\lvert x\rvert-2)$	$(-)(-) = (+)$	$(+)(-) = (-)$	$(+)(+) = (+)$

Hence, from the table $|x| \in (1, 2)$ or

$$1 < |x| < 2 \quad\Leftrightarrow\quad 1 < |x| \text{ and } |x| < 2 \quad\Leftrightarrow\quad x \in (-\infty, -1) \cup (1, \infty) \text{ and } x \in (-2, 2).$$

That is, the solution set is $\{(-\infty, -1) \cup (1, \infty)\} \cap (-2, 2) = (-2, -1) \cup (1, 2)$. ■

4. A math class counts the midterm exam scores as 1/3 of the grade and the final exam scores as 2/3 of the grade. Paul scored 48% on the midterm.

To get an A, Paul must have the total score between 90% and 100% inclusive;
to get a B, Paul must have the total score between 80% and 89% inclusive;
to get a C, Paul must have the total score between 70% and 79% inclusive.

Suppose that all scores with decimal digits get rounded up to the closest higher integers.

(a) What range of scores (in %) on the final exam would make Paul get a C?

(b) What is the highest grade that Paul could get for this class?

Solution: (a) $[81, 94.5]$. (b) B

(a) Let x and y be the midterm and final scores out of 100/3 and 200/3, respectively. Then

$$\frac{x}{100/3} = \frac{48}{100} \qquad \Rightarrow \quad x = 16.$$

To get a C,

$$16 + y \in [70, 79] \quad \Rightarrow \quad 70 \leq 16 + y \leq 79 \quad \Rightarrow \quad 54 \leq y \leq 63.$$

Let Y be the final exam score in %. Then $\frac{Y}{100} = \frac{y}{200/3}$ or $y = \frac{2}{3}Y$. That is, Paul will get a C if

$$54 \leq \frac{2}{3}Y \leq 63 \quad \Rightarrow \quad 81 \leq Y \leq 94.5.$$

(b) Since the maximum of the final score is $y = 200/3$, the maximum score for the grade is

$$\frac{200}{3}\% + 16\% = 82.6666...\% \ \in \ [80, 89].$$

That is, the highest grade that Paul could get for this class is B.

■

5. The weekly demand (the number bought by consumers) for the a product is given by the formula

$$d = 9000 - 60p$$

where p is the price each in dollars.

(a)What is the demand when the price is 30 each?

(b) In what price range will the demand be above 6000?

6. (Optional) A store's revenue $R(x)$ (in Baht) on the sale of x cupcakes is determined by the formula $R(x) = 50x - x^2$. The cost $C(x)$(in Baht) for producing x cupcakes is given by the formula $C(x) = 2x + 400$. For what values of x is the store's profit positive? Note: profit = revenue - cost.

1 Additional Problems

1. Find the solution set for each of the following inequities.

   (a) $\frac{x-1}{x+1} \leq 0$
   Ans: $(-1, 1]$

   (b) $\frac{-x^2-x}{2-x} \geq 0$
   Ans: $[-1, 0] \cup (2, \infty)$

   (c) $\frac{2x}{x-2} - \frac{3x}{x-4} \leq 1$
   Ans: $(-\infty, 2) \cup (4, \infty)$

   (d) $\frac{(2x^2+4x+7)(x-1)}{x^3-5x^2+x-5} \leq 0$
   Ans: $[1, 5)$

   (e) $\frac{x^2+8}{x^2+x+6} \geq 0$
   Ans: $(-\infty, \infty)$

   (f) $\frac{x^2-2x-4}{x^2-x-6} \geq 0$
   Ans: $(-\infty, -2) \cup [1-\sqrt{5}, 3) \cup (1+\sqrt{5}, \infty)$

   (g) $\frac{2x-x^2-3}{x-2x^2-1} \leq 1$
   Ans: $(-\infty, -2] \cup [1, \infty)$

   (h) $\frac{(-x^2+5x-7)(2x+1)}{x^4-1} \geq 0$
   Ans: $(-\infty, -1) \cup [-1/2, 1)$

   (i) $\frac{x^2-1}{2-3x+x^2} > \frac{1}{x}$
   Ans: $(-\infty, 0) \cup (2, \infty)$

   (j) $\frac{x^4+6x^2}{1-2x} \leq x^2$
   Ans: $\{0\} \cup (1/2, \infty)$

   (k) $x^5 + 3x^4 - 23x^3 - 51x^2 + 94x + 120 \geq 0$

   (l) $\frac{\ln(x^2+x+4)}{|\sin(x)-\cos(x)|-3} < 0$
   Ans: $\mathbb{R}$ (Hint: $x^2 + x + 4 > 1$)

2. (Optional) Find the solution set for each of the following inequities.

   (a) $|x-1| < 7$
   Ans: $(-6, 8)$

   (b) $3 < |x+2| < 4$
   Ans: $(-6, -5) \cup (1, 2)$

   (c) $|2x+3| \leq 2$
   Ans: $[-5/2, -1/2]$

   (d) $|3x-1| \geq 1$
   Ans: $(-\infty, 0] \cup [2/3, \infty)$

(e) $\frac{|x+2|}{2} > |x|$
Ans: $(-2/3, 2)$

(f) $|x+1| - |2x-1| < 0$
Ans: $(-\infty, 0) \cup (2, \infty)$

(g) $|2-x| - x \leq 0$
Ans: $[1, \infty)$

(h) $|x+1| > 3 - |x|$
Ans: $(-\infty, -2) \cup (1, \infty)$

(i) $x^2 - 2 \geq \frac{1}{2}|x-1|$
Ans: $(-\infty, \frac{-1-\sqrt{41}}{4}) \cup (3/2, \infty)$

(j) $|x+3| \geq 2 - x$
Ans: $[-1/2, \infty)$

(k) $|2x^2 + x - 1| \geq 2$
Ans: $(-\infty, -3/2] \cup [1, \infty)$

(l) $\frac{|x|}{2} + 3 > x^2$
Ans: $(-2, 2)$

(m) $|x| > \frac{2}{|x+1|}$
Ans: $(-\infty, -2) \cup (1, \infty)$

(n) $\frac{x^2}{x+1} < |x|$
Ans: $(-\infty, -1) \cup (-1/2, 0) \cup (0, \infty)$

(o) $\frac{4}{|x|} \leq \frac{1}{|x|}$
Ans: $\emptyset$

(p) $|x^2 + 1| < |x+1|$
Ans: $(0, 1)$

(q) $(x^2 + 1)(|x+2| - |x|) \geq 0$
Ans: $[-1, \infty)$

(r) $\frac{|1-x|}{x^3 + 2x^2 + 5x + 4} \leq 0$
Ans: $(-\infty, -1) \cup \{1\}$ (Hint: for $f(x) = x^3 + 2x^2 + 5x + 4,\ f(-1) = 0$)