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**EE 320 Introductory Mathematical Economics**  
**Semester 1/2017**

**Practice Problem Set 4 - Answers**

**Single-variable Unconstrained Optimization**

**Question 1 Total product function**

Suppose that  $TP(L) = 120L^2 - L^3$ , where TP is the total output. Answer the following questions

- a. Find the expression for AP and MP.

$$AP(L) = 120L - L^2 \text{ and } MP(L) = 240L - 3L^2$$

- b. Find the value of L that maximizes AP, MP and TP, respectively.

$$\text{Maximize AP} \Rightarrow FOC : AP'(L) = 120 - 2L = 0 \implies L = 60$$

$$SOC : AP''(60) = -2 < 0$$

$$\text{Maximize MP} \Rightarrow FOC : MP'(L) = 240 - 6L = 0 \implies L = 40$$

$$SOC : (MP''(40) = -6 < 0)$$

Maximize TP  $\Rightarrow$

$$FOC : TP'(L) = MP(L) = 240L - 3L^2 = 0 \implies L = 0, 80.$$

$$SOC : (TP''(L) = 240 - 6L \implies TP''(80) < 0.$$

- c. Define the domain set of L that justifies the production function.

One should be noted that (i) L must be greater than or equal to zero. (negative L doesn't make sense. Second, firm will not produce in the region that L is greater than 80 units. When L exceeds 80 units, TP decreases.

## Question 2

Let the total cost function be:

$$TC(Q) = 2Q^2 - 8Q + 10.$$

a) Determine whether  $TC(Q)$  is a convex or concave function.

Ans.  $TC'(Q) = 4Q - 8. \Rightarrow TC''(Q) = 4 > 0$ . Thus,  $TC(Q)$  is a convex function.

b) Find the quantity  $Q^*$  that minimizes the total cost.

Ans.  $TC'(Q) = 4Q - 8 = 0. \Rightarrow Q^* = 2$ .

c) Verify that  $TC(Q^*)$  is the lowest cost by using the second derivative test.

Ans.  $TC''(2) = 4 > 0$ . Thus,  $TC(2) = 2$  is the minimum.

## Question 3

Let the demand function be given by  $Q = 60 - 2P$  where  $P$  is the unit price and  $Q$  is the amount of quantity. Assume that the production function of a firm is given by  $Q = 5\sqrt{L}$  where  $L$  is the number of workers hired. Consider the following questions.

a. Suppose that wage is \$50 per each worker hired. This is fixed, regardless to how many workers hired by a firm. Write down the equation that summarizes the total cost of hiring labor by “L” workers.

$$Cost = W * L = 50L$$

b. Normally, our cost function is written in terms of  $Q$ . Based on “a”, can you find a way to represent the total cost equation in the terms of  $Q$ , rather than  $L$ ?

$$\text{Note that } Q = 5\sqrt{L} \Rightarrow Q^2 = 25L \Rightarrow L = \frac{1}{25}Q^2 .$$

$$Cost = 50L = 50\left(\frac{1}{25}Q^2\right) = 2Q^2$$

c. Use the cost function in “b”, and solve for the level of profit-maximizing output/price. Verify your answer

First, rewrite demand into p-equal form. That is,  $P = \frac{60-Q}{2}$

$$\pi(Q) = \frac{60-Q}{2}Q - 2Q^2$$

$$\text{FOC: } \pi'(Q) = 30 - Q - 4Q = 0$$

$$Q = 6.$$

SOC:  $\pi''(Q) = -5 < 0 \Rightarrow$  This warrants that  $Q = 6$  is actually the profit-maximizing level of output.

- d. At the level of profit-maximizing output, how many workers should the firm hire?

$$Q = 6 \implies L = \frac{1}{25}(6)^2 = 1.44 \text{ units.}$$

#### **Question 4** *Deriving the market supply*

Suppose the cost function of a representative firm can be given by:

$C(Q) = 3Q^2 + 5Q + 75$  where  $Q$  is the level of output produced. Answer the following question

- a. Derive the expression for MC, AVC, and ATC.

$$MC(Q) = 6Q + 5$$

$$AVC(Q) = \frac{TVC}{Q} = 3Q + 5$$

$$ATC(Q) = 3Q + 5 + \frac{75}{Q}$$

- b. Find the level of  $Q$  that results in the lowest total cost.

That is, when  $MC = 0$ .

$$MC(Q) = 6Q + 5 = 0 \implies Q = -5/6.$$

However, it is not possible to have negative  $Q$ .

As you can see from the equation,  $MC$  is always positive. That means,  $C(Q)$  is a strictly increasing function for positive  $Q$ .

As a result, the lowest cost would be when  $Q = 0$ . That's firm only pays for the fixed cost.

- c. Find the supply equation of the representative firm. Make sure you specify the range of price that justifies your equation as the one representing the supply equation.

$$P = MC(Q)$$

$$P = 6Q + 5$$

When  $Q = 0$ ,  $P = 5$ . That is, firm needs to earn at least \$5 if it would stay active in the market.

d. If there are 60 identical firms in the market, derive the market supply curve.

$$Q = \frac{P-5}{6} \implies Q^s_M = 60 * \left(\frac{P-5}{6}\right) = 10(P - 5) \text{ when } P \text{ is greater than or equal to } 5.$$
$$P = 5 + \frac{Q}{10}$$

### Question 5

Given the following function

$$f(x) = 2x^3 + 8x^2 - 32x - 50$$

a. Find the critical value(s) of  $x$  and the corresponding stationary value(s) of  $f(x)$ .

$$\text{Ans. } f'(x) = 6x^2 + 16x - 32 = 0 \implies x^* = -4, 4/3$$

b. Evaluate whether the stationary value(s) found in part a) are relative maxima or minima or inflection points by using *the first-derivative test*.

$$\text{Ans. } f(-4) \text{ is a maximum because } f''(-4)=38 > 0 \text{ and } f''(4/3)=-26 < 0.$$

$$f(4/3) \text{ is a minimum because } f''(1)=-10 < 0 \text{ and } f''(2)=24 > 0.$$

### Question 6

A competitive firm receives a price  $p$  for each unit of its output, and pays a price  $w$  for each unit of its only variable input. It also incurs set up costs of  $F$ . Its output from using  $x$  units of variable input is  $f(x) = \sqrt{x}$ . Determine the firm's revenue, cost, and profit functions.

$$\text{Ans. For } x > 0, R = p\sqrt{x}; C = wx + F; \pi(x) = p\sqrt{x} - wx - F.$$

$$\pi'(x) = 0 \text{ when } w = p/2\sqrt{x} \implies x = p^2/4w^2. \text{ Check: } \pi''(x) < 0$$

### Question 7

The price a firm obtains for a commodity varies with demand  $Q$  according to the formula  $P(Q) = 18 - 0.006Q$ . Total cost is  $C(Q) = 0.004Q^2 + 4Q + 4500$ .

a. Find the firm's profit and the value  $Q$  which maximizes profit.

$$\text{Ans. } \pi(Q) = Q \cdot P(Q) - C(Q) = -0.01Q^2 + 14Q - 4500 \Rightarrow Q^* = 700$$

b. Find a formula for the elasticity of  $P(Q)$  w.r.t.  $Q$ , and find the particular value  $Q^*$  of  $Q$  at which the elasticity is equal to -1.

Ans. This is the price elasticity w.r.t. demand ( $Q=Q_d$ ):

$$E_d = \frac{Q}{P} \frac{dP}{dQ} = \frac{Q}{18-0.006Q} (-0.006) = -1 \Rightarrow Q^* = 1500$$

c. Show that the marginal revenue is 0 at  $Q^*$ .

$$\text{Ans. } MR = 18 - 0.012Q \Rightarrow \text{at } Q^* = 1500, MR = 0.$$

### **Question 8** Monopoly and Subsidy program

A monopolist firm faces the market demand equation given by  $P = 150 - 0.5Q$  and operates under a technology with cost function given by

$TC = 100 + 3Q + 7Q^2$ . Consider the following questions

a. By using the derivative method, find the level of profit-maximizing output and price. Verify that your answer is the correction solution that results in maximized profit.

$$MR = 150 - Q$$

$$MC = 3 + 14Q$$

$$\text{FOC: } MR = MC$$

$$150 - Q = 3 + 14Q$$

$$Q = \frac{147}{15}$$

SOC:

$$\pi'' = MR' - MC' < 0$$

$$MR' = -1$$

$$MC' = 14$$

$$\pi'' = -15 < 0$$

So,  $Q = \frac{147}{15}$  is the level of profit-maximizing output.

Continue with all the information given above, but now add another assumption to the questions. That is, we now assume that government subsidizes the monopolist for \$3 for each unit of output.

- b. Write the cost function of the monopolist when subsidization is taken into account.

$$TC = 100 + 3Q + 7Q^2 - 3Q = 100 + 7Q^2.$$

- c. Find the level of profit-maximizing output and price under the subsidy program.

Redo the same as in (a), but now we use the new TC as obtained in (b).

FOC:

$$MR = MC$$

$$150 - Q = 14Q$$

$$Q = 10.$$

SOC.  $\pi'' = -15 < 0$ . Thus,  $Q = 10$  is the level of profit-maximizing output.

Given this Q, price would be \$145. (This can be obtained by plugging Q into the demand equation.)

### Question 9

A monopolist is operating under a cost function given by,

$$TC = 2Q^3 - 12Q^2 + 94Q + 50,$$

where Q is the quantity of output.

Suppose that the market demand for the goods produced by this monopolist is given by:  $P = 81 - Q^2$ , where P is the price per unit of output.

Consider the following problems.

- a. Determine the *absolute* value of the price elasticity of demand when P = \$54.

$$\text{abs(elasticity)} = 1$$

- b. Without solving for the optimization, determine the revenue-maximizing level of output.

Where elasticity of demand is equal to 1. That is,  $Q = \sqrt{27}$

- c. Derive the profit function, and solve for the profit-maximizing level of output. Verify your answer by using the second-order derivative.

$$\text{profit} = -3Q^3 + 12Q^2 - 12Q - 50$$

$$Q = 2; \pi'' = -12 < 0$$

- d. Would the monopolist change the level of output if government taxes the monopolist based on the profit level that he earns?

Output will remain the same.