B.E. International Program

Faculty of Economics, Thammasat University

EE 320 Introductory Mathematical Economics (Section 046402)

Semester 1/2013

Practice Problem 9
(Optimization with Equality Constraints) [1]

1. a) A firm produces two different kinds A and B of a commodity. The daily cost of producing x units of A and y units of B is

$$C(x,y) = 2x^2 - 4xy + 4y^2 - 40x - 20y + 514$$

Suppose that the firm sells all its output at a price per unit of \$24 for A and \$12 for B. Find the daily production levels of x and y that maximize profit.

b) The firm is required to produce exactly 54 units per day of the two kinds combined. What now is the optimal production plan?

2. Suppose that the utility maximization problem is

$$\max U = xyz \text{ subject to } x + 3y + 4z = 108.$$

a) Solve the utility maximization problem by making U a function of y and z by eliminating the variable x.

b) Solve the above problem by using the Lagrange multiplier method.

3. The profit obtained by a firm from producing and selling x and y units of two brands of a commodity is given by

[1] All questions are from Sydsaeter and Hammond, 2008.

$$P(x, y) = -0.1x^2 - 0.2xy - 0.2y^2 + 47x + 48y - 600$$

a) Find the production levels that maximize profits.

b) A key raw material is rationed so that total production must be restricted to 200 units. Find the production levels that now maximize profits.

4. Use the Lagrange's method to solve the problem

$$\min_{Q_1,Q_2} -40Q_1 + Q_1^2 - 2Q_1Q_2 - 20Q_2 + Q_2^2 \quad \text{Subject to} \quad Q_1 + Q_2 = 15.$$

Use the bordered Hessian to verify that the second-order sufficient conditions are satisfied.

5. A person has utility function

$$u(x, y) = 100xy + x + 2y.$$

Suppose that the price per unit of x is \$2, and that the price per unit of y is \$4. The person receives \$1000 that all has to be spent on the two commodities *x* and *y*. Solve the utility maximization problem, and verify that the second-order sufficient conditions are satisfied.

6. An individual has a Cobb-Douglas utility function$U(m, l) = Am^a l^b$, where *m* is income and *l* is leisure. Here *A*, *a*, and *b* are positive constants, with $a + b \le 1$. A total of T_0 hours are allocated between work *W* and leisure *l*, so that $w + l = T_0$. If the hourly wage is w, then $m = wW$, and the individual's problem is

$$\max_{m,l} Am^a l^b \qquad \text{subject to } \left(\frac{m}{w}\right) + l = T_0.$$

a) Solve the problem by using substitution method.

b) Solve the problem by using the Lagrange method.

7. A firm produces and sells two commodities. By selling *x* tons of the first commodity the firm gets a price per ton given by $p = 96 - 4x$. By selling *y* tons of the other commodity the price per ton is given by $q = 84 - 2y$. The cost of producing and selling *x* tons of the first commodity and *y* tons of the second is given by $C(x, y) = 2x^2 + 2xy + y^2$.

a) Show that the firm's profit function is $P(x, y) = -6x^2 - 3y^2 - 2xy + 96x + 84y$.

b) Compute the first-order partial derivatives of *P*, and find its only stationary point.

c) Suppose the production causes pollution, and that the authorities for this reason require the firm the produce only 11 tons in total. Solve the firm's maximization problem in this case. Verify that the production restrictions do reduce the maximum possible value of *P(x,y)*.

8. a) Solve the utility maximization problem

$$\max_{x,y} U(x,y) = \sqrt{x} + y \qquad \text{subject to} \qquad x + 4y = 100$$

using the Lagrange method, i.e. find the quantities demanded of the two goods.

b) Suppose income increases from 100 to 101. What is the exact increase in the optimal value of U(x, y)? Compare with the value found in (a) for the Lagrange multiplier.

c) Suppose we change the budget constraint to $px + qy = m$, but keep the same utility function. Derive the quantities demanded of the two goods if $m > q^2/4p$.

9. a) Consider the consume demand problem

$$\max_{x,y}[U(x,y) = \alpha \ln (x - a) + \beta \ln (y - b)] \quad \text{subject to} \quad px + qy = m \quad (*)$$

Where α, β, a, b, p, q, and m are positive constants with α + β = 1, and moreover, with $m > ap + bq$. Show that if x*, y* solve the problem (*), then expenditure on the two goods is given by the two linear functions

$$px^* = \alpha m + pa - \alpha(pa + qb); \; qy^* = \beta m + qb - \beta(pa + qb) \qquad (**)$$

of the variables (m, p, q).

b) Let $U^*(p,q,m) = U(x^*,y^*)$ denote the indirect utility function. Show that $\partial U^*/\partial m > 0$ and verify the so-called Roy's identities, $\frac{\partial U^*}{\partial p} = -\frac{\partial U^*}{\partial m} x^*$ and $\frac{\partial U^*}{\partial q} = -\frac{\partial U^*}{\partial m} y^*$.