Vector spaces and subspaces

- Vectors and Vector equations
- Subspaces of R^n
- Null spaces, Column spaces
- Linear independence, Spanning sets
- Basis and Dimension
- Rank
- Linear transformations, change of basis

Vector: A matrix with only one column.

Vectors in $\mathbf{R}^n$ (vectors with n entries):

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

Geometric Description of $\mathbf{R}^2$

Vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the point (x_1, x_2) in the plane.

$\mathbf{R}^2$ is the set of all points in the plane.

Parallelogram rule for addition of two vectors:
If $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^2$ are represented as points in the plane, then $\mathbf{u}+\mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are $\mathbf{0}$, $\mathbf{u}$ and $\mathbf{v}$. (Note that $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.)

EXAMPLE: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Graphs of $\mathbf{u}, \mathbf{v}$ and $\mathbf{u}+\mathbf{v}$ are given below:

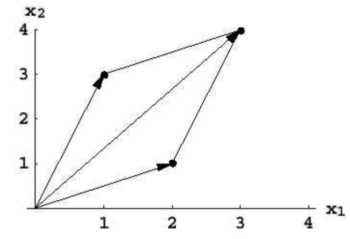
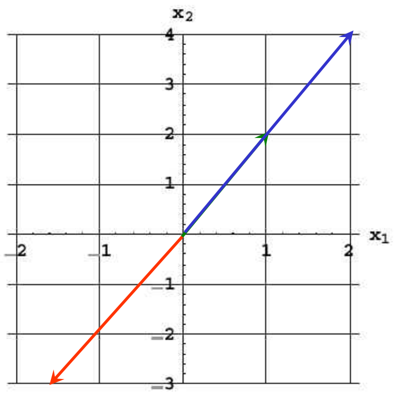


Illustration of the Parallelogram Rule

EXAMPLE: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Express $\mathbf{u}$, $2\mathbf{u}$, and $\frac{-3}{2}\mathbf{u}$ on a graph.



in R^2

Linear Combinations

DEFINITION

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbf{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

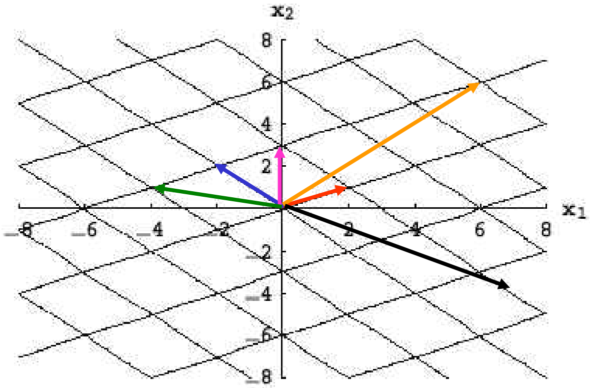
is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ using weights $c_1, c_2, \ldots, c_p$.

Examples of linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$3\mathbf{v}_1 + 2\mathbf{v}_2, \qquad \tfrac{1}{3}\mathbf{v}_1, \qquad \mathbf{v}_1 - 2\mathbf{v}_2, \qquad \mathbf{0}$$

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Express each of the following as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$:

$$\mathbf{a} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}, \mathbf{d} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$$



EXAMPLE: Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 4 \\ 2 \\ 14 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 3 \\ 6 \\ 10 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ 8 \\ -5 \end{bmatrix}$.

Determine if $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$.

Solution: Vector $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ if can we find weights x_1, x_2, x_3 such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}.$$

Vector Equation (fill-in):

Corresponding System:

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= -1 \\ 2x_2 + 6x_3 &= 8 \\ 3x_1 + 14x_2 + 10x_3 &= -5 \end{aligned}$$

Corresponding Augmented Matrix:

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = \_\_\_ \\ x_2 = \_\_\_ \\ x_3 = \_\_\_ \end{array}$$

Review of the last example: $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{b}$ are columns of the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & -1 \\ 0 & 2 & 6 & 8 \\ 3 & 14 & 10 & -5 \end{array}\right]$$
$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{array}$$

Solution to

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$$

is found by solving the linear system whose augmented matrix is

$$\left[\begin{array}{ccc|c} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{array}\right].$$

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}.$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ if and only if there is a solution to the linear system corresponding to the augmented matrix.

The span of a set of vectors

Definition

Suppose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are in $\mathbf{R}^n$; then

Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ = set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$.

Stated another way: **Span**$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written as

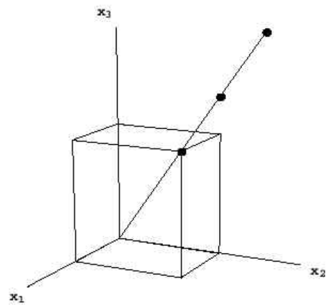
$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p$$

where $x_1, x_2, \ldots, x_p$ are scalars.

A Geometric description of Span {v} and Span {u,v}

The Span of a Set of Vectors

EXAMPLE: Let $\mathbf{v} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$. Label the origin $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ together with $\mathbf{v}$, $2\mathbf{v}$ and $1.5\mathbf{v}$ on the graph below.

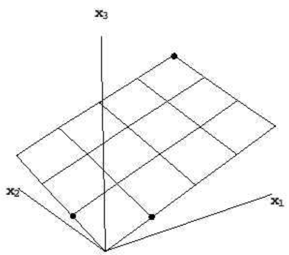


$\mathbf{v}$, $2\mathbf{v}$ and $1.5\mathbf{v}$ all lie on the same line.
Span$\{\mathbf{v}\}$ is the set of all vectors of the form $c\mathbf{v}$.
Here, **Span**$\{\mathbf{v}\}$ = a line through the origin.

- u and v are nonzero vector in R^3 with v is not a multiple of u
- Span {u,v} is a plane in R^3 that contains u ,v and 0
- Span {u,v} contains
  the line in R^3 through u and 0
  and the line in R^3 through v and 0

$\mathbf{u}$, $\mathbf{v}$, $\mathbf{u}+\mathbf{v}$ and $3\mathbf{u}+4\mathbf{v}$ all lie in the same plane.
Span$\{\mathbf{u}, \mathbf{v}\}$ is the set of all vectors of the form $x_1\mathbf{u} + x_2\mathbf{v}$.
Here, **Span**$\{\mathbf{u}, \mathbf{v}\}$ = a plane through the origin.

EXAMPLE: Label $\mathbf{u}$, $\mathbf{v}$, $\mathbf{u}+\mathbf{v}$ and $3\mathbf{u}+4\mathbf{v}$ on the graph below.



EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$.

(a) Find a vector in **Span**$\{\mathbf{v}_1, \mathbf{v}_2\}$.

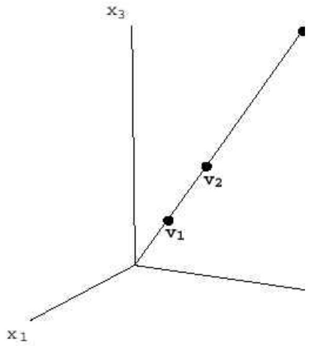
$$x_1\underline{v}_1 + x_2\underline{v}_2 = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Vectors in R^2

(b) Describe **Span**$\{\mathbf{v}_1, \mathbf{v}_2\}$ geometrically.

A line through the origin.

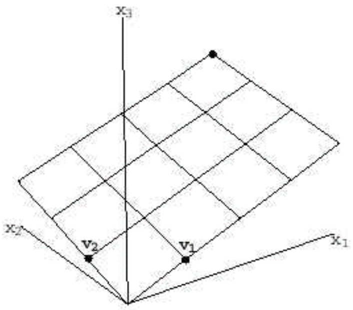
Spanning Sets in $\mathbf{R}^3$



$\mathbf{v}_2$ is a multiple of $\mathbf{v}_1$

Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ =**Span**$\{\mathbf{v}_1\}$ =**Span**$\{\mathbf{v}_2\}$

(line through the origin)



$\mathbf{v}_2$ is **not** a multiple of $\mathbf{v}_1$

Span$\{\mathbf{v}_1, \mathbf{v}_2\}$ =plane through the origin

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix}$. Is **Span**$\{\mathbf{v}_1, \mathbf{v}_2\}$ a line or a plane?

A line $\underline{v}_2 = \frac{3}{2}\underline{v}_1$

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$. Is **b** in the plane spanned by the columns of A?

Solution:

$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 5 \end{bmatrix}$ $\mathbf{b} = \begin{bmatrix} 8 \\ 3 \\ 17 \end{bmatrix}$

Do x_1 and x_2 exist so that

Corresponding augmented matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & 1 & 3 \\ 0 & 5 & 17 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 5 & 17 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 0 & -5 & -21 \\ 0 & 0 & -4 \end{array}\right]$$

So **b** is not in the plane spanned by the columns of A

Is $\underline{b}$ a linear combination of columns of A? If it is $x_1\underline{a}_1 + x_2\underline{a}_2 = \underline{b}$ must have solution.

Vector Spaces and Subspaces

Many concepts concerning vectors in $\mathbf{R}^n$ can be extended to other mathematical systems.

We can think of a *vector space* in general, as a collection of objects that behave as vectors do in $\mathbf{R}^n$. The objects of such a set are called *vectors*.

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms below. The axioms must hold for all **u**, **v** and **w** in V and for all scalars c and d.

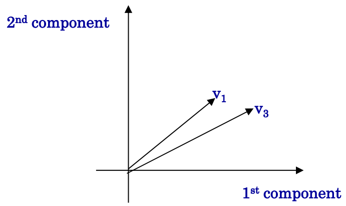
The space R^n consists of all column vectors with n components

A real vector space is a set of vectors together with rules for vector addition and multiplication by real numbers.
The resulting vector must be within the space.

1. $\mathbf{u} + \mathbf{v}$ is in V.
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
4. There is a vector (called the zero vector) $\mathbf{0}$ in V such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is vector $-\mathbf{u}$ in V satisfying $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.
6. $c\mathbf{u}$ is in V.
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$.
9. $(cd)\mathbf{u} = c(d\mathbf{u})$.
10. $1\mathbf{u} = \mathbf{u}$.

Example

R^2 ➔ all 2D real vectors



$$\begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\pi\\e\end{bmatrix}$$

Vector addition?
Scalar multiplication?

Every vector spaces got zero vectors in it.

Not a vector space
Consider all vectors whose components are positive or zero.
If the original space is the x-y plane R^2

Vector addition?

Scalar Multiplication? Multiplying a vector (1,3) by –2➔
This ¼ is not closed under scalar multiplication.

The distinction between a subset and a subspace
✓Can you add vectors? and
✓Can you multiply by scalars without leaving the space?

Vector spaces have to be closed by addition and scalar multiplication.

Subspaces

Vector spaces may be formed from subsets of other vectors spaces. These are called *subspaces*.

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.

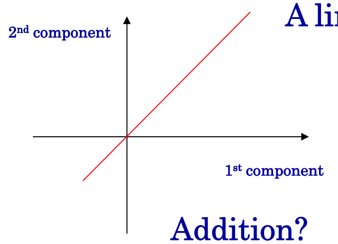
b. For each **u** and **v** are in H, $\mathbf{u}+\mathbf{v}$ is in H. (In this case we say H is closed under vector addition.)

c. For each **u** in H and each scalar c, $c\mathbf{u}$ is in H. (In this case we say H is closed under scalar multiplication.)

If the subset H satisfies these three properties, then H itself is a vector space.

A subspace is a subset which is closed under addition and multiplication.

A vector space inside R^2 ➔ a subspace of R^2



A line through the origin

Addition?

Multiplication?

The zero dimensional space ➔ a subspace contains only one vector, the zero vector. (the smallest subspace)

Vector addition ➔ 0+0=0 ➔ within the subspace

Scalar multiplication➔ C0=0➔ within the subspace

The largest subspace is the whole of the original space.

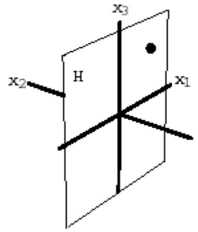
Subspaces of R^2

- R^2 itself
- Any lines through the zero vector (the origin)
- The zero vector

Subspaces of R^3

- R^3 itself
- Any plane through the zero vector (the origin)
- Any line through the zero vector
- The zero vector

EXAMPLE: Let $H = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a \text{ and } b \text{ are real} \right\}$. Show that H is a subspace of $\mathbf{R}^3$.



Solution: Verify properties a, b and c of the definition of a subspace.

a. The zero vector of $\mathbf{R}^3$ is in H (let $a =$ ______ and $b =$ ______).

b. Adding two vectors in H always produces another vector whose second entry is ______ and therefore the sum of two vectors in H is also in H. (H is closed under addition)

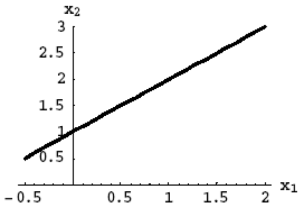
c. Multiplying a vector in H by a scalar produces another vector in H (H is closed under scalar multiplication).

Since properties a, b, and c hold, H is a subspace of $\mathbf{R}^3$.

Note: Vectors $(a, 0, b)$ in H look and act like the points (a, b) in $\mathbf{R}^2$.

EXAMPLE: Is $H = \left\{ \begin{bmatrix} x \\ x+1 \end{bmatrix} : x \text{ is real} \right\}$ a subspace of ?

I.e., does H satisfy properties a, b and c?



Graphical Depiction of H

All three properties must hold in order for H to be a subspace of $\mathbf{R}^2$.

Property (a) is not true because

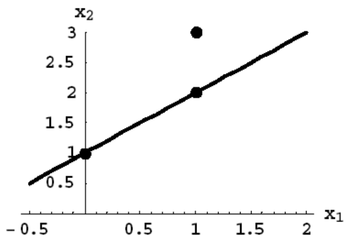
____ Therefore H is not a subspace of $\mathbf{R}^2$.

Another way to show that H is not a subspace of $\mathbf{R}^2$:

Let

$$\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ then } \mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

and so $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, which is _not_ in H. So property (b) fails and so H is not a subspace of $\mathbf{R}^2$.



Property (b) fails

- **A line in R^2 not through the origin is not a subspace of R^2**
- **A plane in R^3 not through the origin is not a subspace of R^3**

$$A=\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}$$ Columns are in_______

The **column space** of an $m \times n$ matrix A (Col A) is the set of all linear combinations of the columns of A.

If $A = [\mathbf{a}_1 \ \dots \ \mathbf{a}_n]$, then

$$\text{Col } A = \text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$$

The column space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^m$.

Column space of A is a subspace of __R^m______
What are in this subspace?

Subspaces are tied directly to matrix A and they give information about the system Ax=b

Connection with linear system *Ax=b*

- Does *Ax=b* have solution for every *b*?

- Which RHS allow this system to be solved?

$$\begin{bmatrix}1&1&2\\2&1&3\\3&1&4\\4&1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix} \longrightarrow u\begin{bmatrix}1\\2\\3\\4\end{bmatrix}+v\begin{bmatrix}1\\1\\1\\1\end{bmatrix}+w\begin{bmatrix}2\\3\\4\\5\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix}$$

Find numbers u,v,w that multiply col1, col2, col3 to produce the vector *b*. The system is solvable exactly when such coefficient exist.

The subset of attainable RHS b is the set of all combinations of the columns of A.

The equations Ax=b can be solved if and only if b lies in *the column space of A*

Recall that if $A\mathbf{x} = \mathbf{b}$, then $\mathbf{b}$ is a linear combination of the columns of A. Therefore

$$\text{Col } A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbf{R}^n\}$$

We can also describe the result geometrically. Ax=b can be solved if and Only if b lines in the plane that is spanned by 2 calcium vectors

EXAMPLE: Find a matrix A such that $W = Col\ A$ where

$$W = \left\{\begin{bmatrix}x-2y\\3y\\x+y\end{bmatrix} : x,y \text{ in } \mathbf{R}\right\}.$$

Solution:

$$\begin{bmatrix}x-2y\\3y\\x+y\end{bmatrix} = x\begin{bmatrix}1\\0\\1\end{bmatrix}+y\begin{bmatrix}-2\\3\\1\end{bmatrix}$$

$$=\begin{bmatrix}\ \ \ \ \ \ \end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Therefore $A = \begin{bmatrix}\ \ \ \ \ \ \end{bmatrix}$.

The column space of an $m \times n$ matrix A is all of $\mathbf{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbf{R}^m$.

The **null space** of an $m \times n$ matrix A, written as Nul A, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\text{Nul } A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbf{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\} \qquad \text{(set notation)}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Null space is in __________

The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbf{R}^n$.

Proof: Nul A is a subset of $\mathbf{R}^n$ since A has n columns. Must verify properties a, b and c of the definition of a subspace.

Property (a) Show that $\mathbf{0}$ is in Nul A. Since _________, $\mathbf{0}$ is in ___________.

Therefore

$A(\mathbf{u}+\mathbf{v})$ = ______ + ______ = ______ + ______ = ___.

Property (b) If $\mathbf{u}$ and $\mathbf{v}$ are in Nul A, show that $\mathbf{u}+\mathbf{v}$ is in Nul A. Since $\mathbf{u}$ and $\mathbf{v}$ are in Nul A,

_________ and _________.

Property (c) If $\mathbf{u}$ is in Nul A and c is a scalar, show that $c\mathbf{u}$ in Nul A:

$$A(c\mathbf{u}) = \_\_\_A(\mathbf{u}) = c\mathbf{0} = \mathbf{0}.$$

Since properties a, b and c hold, A is a subspace of $\mathbf{R}^n$.

Solving $A\mathbf{x} = \mathbf{0}$ yields an ***explicit description*** *of Nul A.*

EXAMPLE: Find an explicit description of Nul A where

$$A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}$$

Solution: Row reduce augmented matrix corresponding to $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 3 & 6 & 6 & 3 & 9 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}$$

Then

Nul A = span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$

The Contrast Between Nul A and Col A

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) The column space of A is a subspace of $\mathbf{R}^k$ where k =_____.

(b) The null space of A is a subspace of $\mathbf{R}^k$ where k =_____.

(c) Find a nonzero vector in Nul A (There are infinitely many possibilities.)

$$\_\_\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + \_\_\begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix} + \_\_\begin{bmatrix} 3 \\ 7 \\ 10 \\ 1 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

(d) Find a nonzero vector in Nul A. Solve $A\mathbf{x} = \mathbf{0}$ and pick one solution.

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 7 & 0 \\ 3 & 6 & 10 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ row reduces to } \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -2x_2$

x_2 is free

$x_3 = 0$

Let x_2 = ___ and then

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

Review

A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.

b. For each **u** and **v** in H, $\mathbf{u} + \mathbf{v}$ is in H. (In this case we say H is closed under vector addition.)

c. For each **u** in H and each scalar c, $c\mathbf{u}$ is in H. (In this case we say H is closed under scalar multiplication.)

If the subset H satisfies these three properties, then H itself is a vector space.

If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.

The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$.

The column space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^m$.

(b) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : \begin{matrix} x - y = 0 \\ y + z = 0 \end{matrix} \right\}$

Solution: Rewrite $\begin{matrix} x - y = 0 \\ y + z = 0 \end{matrix}$ as

$$\begin{bmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So V =Nul A where $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Since Nul A is a subspace of $\mathbf{R}^2$, V is a vector space.

(c) $S = \left\{ \begin{bmatrix} x + y \\ 2x - 3y \\ 3y \end{bmatrix} : x, y, z \text{ are real} \right\}$

One Solution: Since

$$\begin{bmatrix} x + y \\ 2x - 3y \\ 3y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix},$$

S =span$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \right\}$; therefore S is a vector ace by 1.

Another Solution: Since

$$\begin{bmatrix} x + y \\ 2x - 3y \\ 3y \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + y \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix},$$

S =Col A where $A = \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 3 \end{bmatrix}$; therefore S is a vector space, since a column space is a vector space.

Linear Independence

Definition

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in a vector space V is said to be **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution $c_1 = 0, \ldots, c_p = 0$.

Trivial combination, with all weights C_i=0, produces the zero vector.

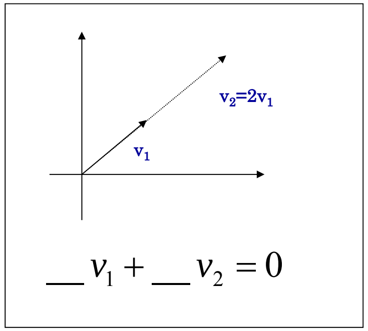
The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists weights $c_1, \ldots, c_p$, not all 0, such that

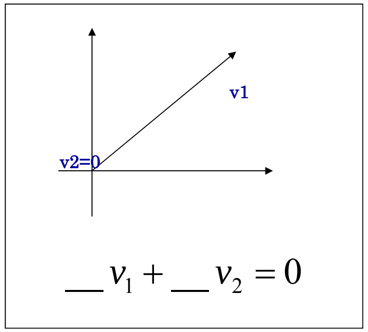
$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}.$$

Vectors $v_1, v_2, \ldots, v_p$ are linear independent of no combination give zero vector (except the zero combination all c=0)

Linear Independence

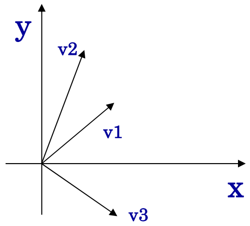
Linearly independence if no combination gives the zero vector except the zero combination.





If one vector is zero dependence is said

Linear Independence



$$AC = \begin{bmatrix} 2 & 1 & 2 \\ 2 & 3 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$$

- 2 vectors are dependent if they lie on the same line.
- 3 vectors are dependent if they lie in the same plane.
- A random choice of vectors, without any special accident, should produce linear independence.
- 4 vectors are always linearly dependent in R^3

Linear Independence

Repeat the definition

When $v_1, v_2 \ldots\ldots v_n$ are columns of *A*

- **They are independent if the null space of A is the zero vector only**
- **They are dependent if there are something in the null space.**

$$\underline{\mathbf{A}}\underline{\mathbf{C}} = 0 \quad \text{for nonzero } \underline{\text{C}}$$

The columns of an *mxn* matrix are independent

- ❑ **All columns are _____ columns**
- ❑ **Rank=_______ ➔the null space of A is only 0 (no free variable)**
- ❑ **Rank<________ ➔at least one free variable**

Linear Independence

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$$

❑What is the combination of column with weights –3, 1,0,0?

❑2× row2- 5 × row1=______

Dependent? or Independent?

The null space of *A* contains only the zero vector➔ the columns of *A* are linearly independent

$$A = \begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

$$c_1 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Dependent? or Independent?

Linear Independence

For any echelon matrix *U* ; the nonzero rows must be ____________

If we pick out the columns that contain the pivots
➔ linearly independent

$$U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Column 1 and 3 are ______
- No set of 3 columns is______
- Column 1 and column 4 are __________

The nonzero rows of an echelon matrix *U* are the rows that contain pivots (r) and are linearly independent.

Linear Independence

The columns of *n by n* identity matrix

$$I = \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & 1 & . & 0 \\ . & . & . & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Dependent? or Independent?

In R^4

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad e_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Coordinate vectors

Linear Independence

To check any set of vectors $v_1, v_2 .... v_n$ for linear independence

- Form the matrix *A* whose n columns are the given vectors.
- Solve for $\underline{\mathbf{AC}} = 0$

Dependent ➔ a solution other than $\underline{C}=0$ (zero vector)

Independent ➔ no free variables (rank is n) ➔ none of vector except $\underline{C}$ =0 is in the null space

m<n ➔ impossible for the columns to be linearly independent!
rank< *n (we cannot have n pivots)*

There cannot be n pivots, since there are not enough rows to hold them

A set of n vectors in R^m must be linearly dependent if n>m

Linear Independence

Example

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$$

Impossible to have 3 independent vectors in R^2

$$A \to U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Spanning a subspace

A set of vectors $w_1, ...\ w_l$ spans a space means the space consists of all linear combinations of those vectors.

General definition

If a vectors space *V* consists of all linear combinations of the particular vectors $w_1, ...w_l$ then these vectors span the space. In other words, every vector *v* in V can be expressed as some combination of the *w's:*

$$v = c_1 w_1 + ... + c_l w_l \quad \text{for some coefficients } c_i$$

The coefficients need not be unique because the spanning set might be excessively large

Spanning a subspace

Example The column space of a matrix is exactly the space that is spanned by its columns.
~ Columns of a matrix span a column s

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in V and let $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

If one of the vectors in S - say $\mathbf{v}_k$ - is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

To decide if b is a combination of the columns ➔ solve $Ax=b$	To decide if the columns are independent ➔ solve $Ax=0$
Spanning involves the column space.	*Independence involves the null space*

MA332 LINEAR ALGEBRA

Spanning a subspace

EXAMPLE: Suppose $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$.

Solution: If $\mathbf{x}$ is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3(\_\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\mathbf{v}_2)$$

$$= \_\_\_\_\_\_\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\_\_\_\_\_\mathbf{v}_2$$

Therefore,

$$\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}.$$

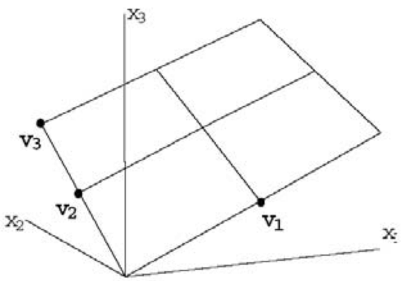
MA332 LINEAR ALGEBRA

A Basis Set

Let H be the plane illustrated below. Which of the following are valid descriptions of H?

(a) $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ (b) $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_3\}$

(c) $H = \text{Span}\{\mathbf{v}_2, \mathbf{v}_3\}$ (d) $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$



MA332 LINEAR ALGEBRA

A Basis Set

A *basis set* is an "efficient" spanning set containing no unnecessary vectors. In this case, we would consider the linearly independent sets $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{v}_1, \mathbf{v}_3\}$ to both be examples of basis sets or bases (plural for basis) for H.

DEFINITION

Let H be a subspace of a vector space V. An indexed set of vectors $\beta = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a basis for H if

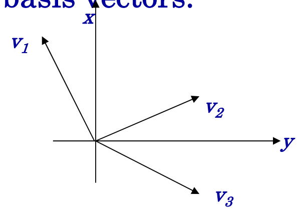
(i) β is a linearly independent set, and

(ii) $H = \text{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$.

In other word ➔*A basis* for a vector space is a set of vectors having two properties at once:
1. They are linearly independent.
2. They span the space.

Basis

- Every vector in the space is a combination of the basis vectors, because they span.
- The combination is unique.
- There is one and only one way to write v as a combination of the basis vectors.



The vector v_1 by itself is __________ but fails to__________ R^2

The vectors v_1, v_2, v_3 span R^2 but are not __________

Any two of these vectors have both properties➔ form a basis.

MA332 LINEAR ALGEBRA

Basis

Example Space in R^3

One basis is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ Are these vectors independent?

The null space of identity matrix is only zero. Therefore columns are______

Another basis $\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\2\\5\end{bmatrix}, \begin{bmatrix}3\\3\\8\end{bmatrix}$

v_1 & v_2 :they are independent but may not span R^3(there can be vector in R^3 that is not combination of those vectors)

If $v_3 = (3,3,7)^T$ ➔ dependent it lies on the same plane as v_1 , v_2
V_3 must be any vector that does not lies in the same plane as v_1 , v_2

How to check it?

Square matrix is invertible ➔ columns are independent →a basis for R^n

MA332 LINEAR ALGEBRA

EXAMPLE: Let $\mathbf{v}_1 = \begin{bmatrix}1\\2\\0\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}1\\0\\3\end{bmatrix}$.

Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a **basis** for $\mathbf{R}^3$?

Solution: Again, let $A = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix} = \begin{bmatrix}1 & 0 & 1\\2 & 1 & 0\\0 & 1 & 3\end{bmatrix}$. Using row reduction,

$$\begin{bmatrix}1 & 0 & 1\\2 & 1 & 0\\0 & 1 & 3\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1\\0 & 1 & -2\\0 & 1 & 3\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1\\0 & 1 & -2\\0 & 0 & 5\end{bmatrix}$$

MA332 LINEAR ALGEBRA

EXAMPLE: Find a basis for Nul A where

$$A = \begin{bmatrix}3 & 6 & 6 & 3 & 9\\6 & 12 & 13 & 0 & 3\end{bmatrix}.$$

Solution: Row reduce $\begin{bmatrix}A & \mathbf{0}\end{bmatrix}$:

$$\begin{bmatrix}1 & 2 & 0 & 13 & 33 & 0\\0 & 0 & 1 & -6 & -15 & 0\end{bmatrix}$$

$x_1 = -2x_2 - 13x_4 - 33x_5$
$x_3 = 6x_4 + 15x_5$
x_2, x_4 and x_5 are free

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}-2x_2 - 13x_4 - 33x_5\\x_2\\6x_4 + 15x_5\\x_4\\x_5\end{bmatrix} = x_2\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix} + x_4\begin{bmatrix}-13\\0\\6\\1\\0\end{bmatrix} + x_5\begin{bmatrix}-33\\0\\15\\0\\1\end{bmatrix}$$

(↑ **u**, ↑ **v**, ↑ **w**)

Basis

Example

$$U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The four columns span the column space but not ________

The columns that contain pivots are a basis for the column space.

The columns of a matrix span its column space.
If they are independent ➔ a basis for the column space

(whether matrix is square or rectangular)

The columns to be a basis for the space R^n,
the matrix must be square and invertible.

Bases for Col A

EXAMPLE: Find a basis for Col A, where

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}.$$

Solution: Row reduce:

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 \end{bmatrix}$$

Note that

$\mathbf{b}_2 = $ ____ $\mathbf{b}_1$ and $\mathbf{a}_2 = $ ____ $\mathbf{a}_1$

$\mathbf{b}_4 = 4\mathbf{b}_1 + 5\mathbf{b}_3$ and $\mathbf{a}_4 = 4\mathbf{a}_1 + 5\mathbf{a}_3$

$\mathbf{b}_1$ and $\mathbf{b}_3$ are not multiples of each other

$\mathbf{a}_1$ and $\mathbf{a}_3$ are not multiples of each other

Elementary row operations on a matrix do not affect the linear dependence relations among the columns of the matrix.

Therefore Span$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ =Span$\{\mathbf{a}_1, \mathbf{a}_3\}$ and $\{\mathbf{a}_1, \mathbf{a}_3\}$ is a basis for Col A.

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ -4 \\ 6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$. Find a basis for Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Solution: Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -4 & 6 \\ -3 & 6 & 9 \end{bmatrix}$ and note that Col A = Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

By row reduction, $A \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore a basis

for Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is $\left\{ \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \begin{bmatrix} \\ \\ \\ \end{bmatrix} \right\}$.

The pivot columns of a matrix *A* form a basis for Col *A*

Review:

1. To find a basis for Nul A, use elementary row operations to transform $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$ to an equivalent reduced row echelon form $\begin{bmatrix} B & \mathbf{0} \end{bmatrix}$. Use the reduced row echelon form to find parametric form of the general solution to $A\mathbf{x} = \mathbf{0}$. The vectors found in this parametric form of the general solution form a basis for Nul A.

2. A basis for Col A is formed from the pivot columns of A. **Warning: Use the pivot columns of A, not the pivot columns of B, where B is in reduced echelon form and is row equivalent to** A.

Dimension of a vector space

A vector space has infinitely many different bases
but something in common → The same number of vectors

The x-y plan → 2 vectors in every basis → dimension=2

R^3 → Number of vectors = 3

R^n → Number of vectors = n

DEFINITION of dimension of a vector space

If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as dim V, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be 0. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

EXAMPLE: The standard basis for $\mathbf{R}^n$ is $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the columns of I_n. So, for example, dim $\mathbf{R}^3 = 3$.

Dimension of a vector space

$$U = \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

What is a basis for the column space of *U*?
Column____and Column_____
which are the __________column.

The column space of *U* has dimension 2
"Two dimensional subspace of R^3"

Example

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

EXAMPLE: Find a basis and the dimension of the subspace

$$W = \left\{ \begin{bmatrix} a+b+2c \\ 2a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{bmatrix} : a,b,c,d \text{ are real} \right\}.$$

Solution: Since
$$\begin{bmatrix} a+b+2c \\ 2a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

W = span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

- Note that $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, so by the Spanning Set Theorem, we may discard $\mathbf{v}_3$.
- $\mathbf{v}_4$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a basis for W.
- Also, dim $W =$____.

EXAMPLE: ***Dimensions of subspaces of R^3***

0-dimensional subspace contains only the zero vector $\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$.

1-dimensional subspaces. Span$\{\mathbf{v}\}$ where $\mathbf{v} \neq \mathbf{0}$ is in $\mathbf{R}^3$.

These subspaces are ______________ through the origin.

2-dimensional subspaces. Span$\{\mathbf{u}, \mathbf{v}\}$ where $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbf{R}^3$ and are not multiples of each other.

These subspaces are ______________ through the origin.

3-dimensional subspaces. Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly independent vectors in $\mathbf{R}^3$. This subspace is $\mathbf{R}^3$ itself because the columns of $A = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$ span $\mathbf{R}^3$ according to the IMT.

Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V.$$

EXAMPLE: Let $H = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$. Then H is a subspace of $\mathbf{R}^3$ and $\dim H < \dim \mathbf{R}^3$.

We could expand the spanning set $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$ to $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ to form a basis for $\mathbf{R}^3$.

THE BASIS THEOREM

Let V be a $p-$ dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p vectors in V is automatically a basis for V. Any set of exactly p vectors that spans V is automatically a basis for V.

Dimensions of Col A and Nul A

Recall our techniques to find basis sets for column spaces and null spaces.

EXAMPLE: Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix}$. Find dim Col A and dim Nul A.

Solution

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

So $\left\{\begin{bmatrix} \quad \\ \quad \end{bmatrix}, \begin{bmatrix} \quad \\ \quad \end{bmatrix}\right\}$ is a basis for Col A and dim Col $A = 2$.

Now solve $A\mathbf{x} = \mathbf{0}$ by row-reducing the corresponding augmented matrix. Then we arrive at

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 2 & 4 & 7 & 8 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$x_1 = -2x_2 - 4x_4$$
$$x_3 = 0$$

and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So $\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis for Nul A and dim Nul $A = 2$.

Dimension of a vector space

Dim Col A = number of pivot columns of A (Rank)
Dim Nul A = number of free variables of A (n-r)

•Any linearly independent set V can be extended to a basis, by adding more vectors if necessary.
•Any spanning set V can be reduced to a basis, by discarding vectors if necessary.

•A basis is a maximal independent set.
It cannot made larger without losing independence
•A basis is also a minimal spanning set.
It cannot be made smaller and still span the space.

~~Basis of a matrix~~
~~Rank of a space~~
~~Dimension of a basis~~

Dimension of the space
Rank of the matrix

Rank

The set of all linear combinations of the row vectors of a matrix A is called the **row space** of A and is denoted by Row A.

EXAMPLE: Let

$$A=\begin{bmatrix}-1 & 2 & 3 & 6\\ 2 & -5 & -6 & -12\\ 1 & -3 & -3 & -6\end{bmatrix} \quad \text{and} \quad \begin{aligned}\mathbf{r}_1 &= (-1,2,3,6)\\ \mathbf{r}_2 &= (2,-5,-6,-12)\\ \mathbf{r}_3 &= (1,-3,-3,-6)\end{aligned}.$$

Row A =Span$\{\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3\}$ (a subspace of $\mathbf{R}^4$)

$\boxed{\text{Col } A^T = \text{Row } A}$.

When we use row operations to reduce matrix A to matrix B, we are taking linear combinations of the rows of A to come up with B. We could reverse this process and use row operations on B to get back to A. Because of this, the row space of A equals the row space of B.

THEOREM 13

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as B.

$$A=\begin{bmatrix}-1 & 2 & 3 & 6\\ 2 & -5 & -6 & -12\\ 1 & -3 & -3 & -6\end{bmatrix} \text{ and } B=\begin{bmatrix}-1 & 2 & 3 & 6\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$$

are row equivalent. Find a basis for row space, column space and null space of A. Also state the dimension of each.

Basis for Row A : $\{ \qquad \}$

dim Row A :_____

Basis for Col A : $\left\{\begin{bmatrix} \\ \\ \end{bmatrix}, \begin{bmatrix} \\ \\ \end{bmatrix}\right\}$

dim Col A :_____

To find Nul A, solve $A\mathbf{x} = \mathbf{0}$ first:

$$\begin{bmatrix} -1 & 2 & 3 & 6 & 0 \\ 2 & -5 & -6 & -12 & 0 \\ 1 & -3 & -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & 6 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & -6 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_3 + 6x_4 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis for Nul A : $\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and dim Nul A = ______

Note the following:

dim Col A = # of pivots of A = # of nonzero rows in B = dim Row A.

dim Nul A = # of free variables = # of nonpivot columns of A.

DEFINITION

The **rank** of A is the dimension of the column space of A.

rank A = dim Col A =# of pivot columns of A = dim Row A.

$$\underbrace{\text{rank } A}_{\updownarrow} + \underbrace{\text{dim Nul } A}_{\updownarrow} = \underbrace{n}_{\updownarrow}$$

$$\left\{\begin{matrix}\text{\# of pivot}\\ \text{columns}\\ \text{of } A\end{matrix}\right\} \quad \left\{\begin{matrix}\text{\# of nonpivot}\\ \text{columns}\\ \text{of } A\end{matrix}\right\} \quad \left\{\begin{matrix}\text{\# of}\\ \text{columns}\\ \text{of } A\end{matrix}\right\}$$

THE RANK THEOREM

The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A + \dim \text{Nul } A = n.$$

Since Row A = Col A^T,

$$\boxed{\text{rank } A = \text{rank } A^T}.$$

EXAMPLE: Suppose that a 5×8 matrix A has rank 5. Find dim Nul A, dim Row A and rank A^T. Is Col $A = \mathbf{R}^5$?

Solution:

$$\underbrace{\text{rank } A}_{\updownarrow\atop 5} + \underbrace{\dim \text{Nul } A}_{\downarrow\atop ?} = \underbrace{n}_{\updownarrow\atop 8}$$

$5 + \dim \text{Nul } A = 8 \quad \Rightarrow \quad \dim \text{Nul } A =$ _____

dim Row A = rank A = _____ $\Rightarrow$ rank A^T = rank _____ = _____

Since rank A = # of pivots in $A = 5$, there is a pivot in every row. So the columns of A span $\mathbf{R}^5$

Hence Col $A = \mathbf{R}^5$.

EXAMPLE: For a 9×12 matrix A, find the smallest possible value of dim Nul A.

Solution:

$$\text{rank } A + \dim \text{Nul } A = 12$$

$$\dim \text{Nul } A = 12 - \underbrace{\text{rank } A}$$

largest possible value=_____

smallest possible value of dim Nul A = _____

Visualizing Row A and Nul A

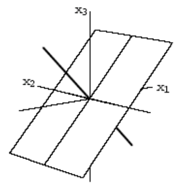
EXAMPLE: Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \end{bmatrix}$. One can easily verify the following:

Basis for Nul $A = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and therefore Nul A is a plane in $\mathbf{R}^3$.

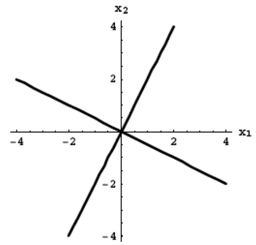
Basis for Row $A = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ and therefore Row A is a line in $\mathbf{R}^3$.

Basis for Col $A = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and therefore Col A is a line in $\mathbf{R}^2$.

Basis for Nul $A^T = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ and therefore Nul A^T is a line in $\mathbf{R}^2$.



Subspaces Nul A and Row A



Subspaces Nul A^T and Col A